TRANSFORMATION OF PAR YIELD CURVES ON ZERO YIELD CURVES WITH FOCUS ON THE SMITH-WILSON METHOD

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Abstract

The paper is about yield curves, their various types, interplays and transformations from one type to another. The primary goal of the paper is an analysis and a comparison of two methods used for creation of zero yield curves: Smith-Wilson method and bootstrapping method. The paper explains their pros and cons including their usual application. The paper focuses on the Smith-Wilson method and its main benefits for the yield curves modelling also considering an impact of ultimate forward rates. The method helps to transform par yield rates, that belong to bonds and swaps which are liquid and available at the market, to risk-free zero yield rates. We empirically tested its advantages in the paper on the data collected from Bloomberg database. Our practical analysis covered yield curves of the following currencies: CZK, USD, EUR, CHF, JPY and GBP. In the last chapter, we analysed the Covid-19 pandemic impact on the 6 tested yield curves.

Key words: yield curves, Smith-Wilson method, bootstrapping

JEL Code: C53, C55, G17

Introduction

The paper is about yield curves, their various types, interplays and transformations from one type to another. There are three main types of yield curves: par, zero and forward yield curves. The primary goal of the paper is an analysis and a comparison of two methods used for creation of zero yield curves: Smith-Wilson method and bootstrapping method. The paper explains their pros and cons including their usual application.

The paper focuses on the Smith-Wilson method and its main benefits for the yield curves modelling also considering a macroeconomic perspective regarding the yield curves environment and calculating an impact of ultimate forward rates (Smith and Wilson, 2000). The method helps to transform par yield rates, that belong to bonds and swaps which are liquid and available at the market, to risk-free zero yield rates. Par rates may contain coupons or credit spreads and don't lead directly to a creation of one continuous yield curve. On the other hand, risk-free zero rates form a continuous curve that is suitable for discounting of assets and liabilities, especially in insurance and banking sectors (Bohn, Elkenbracht-Huizing and Adler, 2014), (Christensen, Lopez and Rudebusch, 2015). Smith-Wilson method enables us to calculate a zero yield rate at each maturity without over-dependency on linear interpolation and extrapolation.

We empirically tested Smith Wilson's advantages in the paper on the data collected from Bloomberg terminal (Bloomberg L.P., 2021) and software used throughout the paper is Python and MS Excel. Our practical analysis covered yield curves of the following currencies: CZK, USD, EUR, CHF, JPY and GBP. For each of them, we transformed the par rates into the zero yield curves for three reporting dates:

- 31.1.2020 (a while before the Covid-19 pandemic outbreak)
- 30.4.2020 (a while after the Covid-19 pandemic outbreak)
- 30.4.2021 (more than one year after the Covid-19 pandemic outbreak)

In the last chapter, we analysed the Covid-19 pandemic impact on the 6 tested yield curves. The Covid-19 pandemic presents a kind of stress period and we tried to test Smith Wilson's method and its stability and reasonability in times of market distress.

1 Yield curves and their types

First of all, we start with a definition of a yield curve. A yield curve is a line that joins yields of bonds (rewards paid for bonds' holding) which have the same credit quality but which differ in maturity dates. The yield curve thus plots the relationship between the yield (maybe be expressed as an interest rate) and the maturity of financial instruments (Choudhry, 2019).

The yield curves play an important role in the economic and financial analysis as they help us to predict future interest rate changes, to manage profitability and to monitor the economic activity. The yield curves may be described by three main visual characteristics: level, slope and shape

The level helps us to cluster countries and yield curves into various categories with regards to their different profitability and risk expectations. The level is usually influenced by credit quality of underlying financial instruments. The slope and shape help us with future predictions. A normal yield curve is upward sloping curve. The longer the time to maturity of a bond (higher tenors of a yield curve), the higher the credit risk and the lower the investor's flexibility to withdraw money back. Therefore, these cons are compensated by a higher reward/yield for higher tenors of a yield curve. However, there are occasionally also downward sloping and flat curves. Both types of these unusual curves reflect some specific conditions of certain countries or companies.

There are three main types of yield curves: par yield curves (or just par rates), zero (or spot) yield curves and forward yield curves. A par yield curve is a yield curve connecting all par rates that are currently traded at the market. The par curve is not continuous and may contain just a few par rates. Hence, it contains a precise information on the market developments, but it is not complete to capture all needed maturities and tenors that may be required for discounting in internal models. A zero yield curve is a continuous yield curve fitting all available par rates and considering (in some cases) information about the future. Forward curves are special curves that indicate what a yield will be if we make a deal at some future point in time. Therefore, they include lots of insights about future developments. The most useful for insurance and banking internal models and pricing are zero (or spot) yield curves that try to absorb as much information on the market and also on the future as available. Crucial is to derive zero yield curves from available par and ultimate forward rates (Rebonato, 2015).

2 Bootstrapping method

The bootstrapping method is used for a construction of zero yield curves from the prices and coupons (par rates) of bonds and swaps. The method can thus indirectly transform par rates into a zero yield curve using also prices of used bonds or swaps. The bootstrapping can handle various coupon payment frequencies and can be based on zero-coupon bonds as well. A high coupon on a bond doesn't guarantee a high real yield as the purchasing bond price can be very high. The derived zero yield curve is therefore very convenient as it enables us to compare real zero/spot yield of a diverse set of coupon-bearing financial instruments which is the main advantage of the method. The bootstrapping is based on an iterative process called as a forward substitution. The forward substitution is expressed by the following formula

$$1 = C_1 * \Delta_1 * df_1 + C_2 * \Delta_2 * df_2 + C_3 * \Delta_3 * df_3 + \cdots + (1 + C_n * \Delta_n) * df_n, \quad (1)$$

where df_n are bootstrapping discount factors:

$$df_n = \frac{(1 - \sum_{i=1}^{n-1} C_n \Delta_i df_i)}{(1 + C_n \Delta_n)}, \quad (2)$$

where C_n is a coupon of the n year bond, Δ_i is a length of the time period, df_i represents a time period discount factor and df_n is a discount factor for the entire period (Hagan and West, 2006). The bootstrapping method has some disadvantages too. For example, it is dependent on market availability of bonds' present values. Moreover, the method requires to calculate all missing tenors of the par yield curve using linear (or other) interpolation. In addition to that the method doesn't contain any perspective on the long-term developments of the yield curve (eg. the UFR) (Siegel and Nelson, 1988). Longer tenors may be calculated just by a simple (and thus less accurate) extrapolation. We have prepared an example of the bootstrapping method in Tab.1

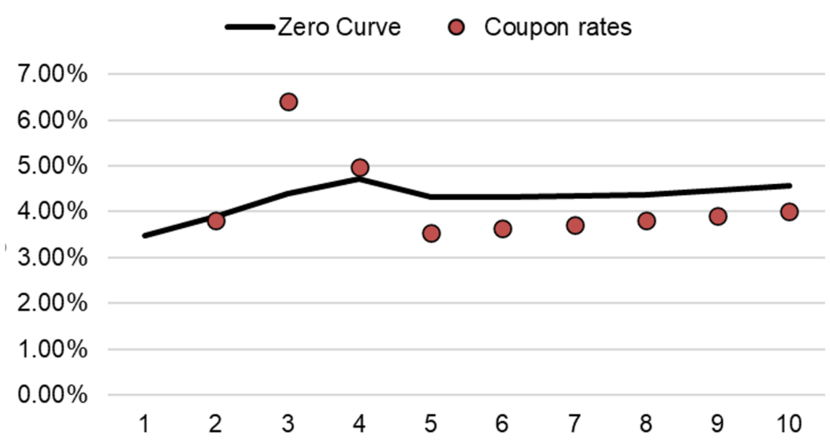
Tab. 1: Bootstrapping of 10 various bonds with their coupons and ask prices

Bond Number	Coupon %	Ask Price	Discount factor	Spot/Zero Rate	Coupon rate
1	0.00	96.7	0.9665	3.47%	0.00%
2	3.80	99.9	0.9266	3.89%	3.80%
3	6.40	105.7	0.8793	4.38%	6.40%
4	4.98	101.1	0.8320	4.70%	4.98%
5	3.55	96.6	0.8093	4.32%	3.55%
6	3.63	96.4	0.7756	4.33%	3.63%
7	3.71	96.3	0.7426	4.34%	3.71%
8	3.80	96.3	0.7103	4.37%	3.80%
9	3.90	96.0	0.6750	4.46%	3.90%
10	4.00	95.8	0.6397	4.57%	4.00%

Source: Bloomberg Terminal data, authors' own elaboration, 1000 bootstrapping replications used

The example covers 10 various bonds with their coupons and ask prices. The output of the bootstrapping is the spot/zero yield curves. In brief, the bootstrapping's main contribution is a standardization of various underlying instruments with different structures of cash-flows into one zero curve.

Fig. 2: Time to maturity in years (Tenors)



Source: Bloomberg Terminal data, authors' own elaboration

3 Smith-Wilson Method

In addition to the bootstrapping's contributions, the Smith-Wilson methods brings a more comprehensive approach to yield curves' transformations. The Smith-Wilson method is built on a set of pricing function, one for each time step to maturity. The SW pricing functions follow the formula:

$$P(t) = e^{-UFR*t} + \sum_{j=1}^{N} \xi_j * W(t, u_j), \quad where \ t \geq 0 \tag{3}$$

and the W describes kernel functions calculated as follows

$$W(t, u_j) = e^{-UFR*(t+u_j)} * \{\alpha * \min(t, u_j) - 0.5 * e^{-\alpha.\max(t,u_j)} * \left(e^{\alpha*\min(t,u_j)} - e^{-\alpha*\min(t,u_j)}\right)\}, \tag{4}$$

where ξ_j are parameters used for fitting the actual yield curve, α mean reversion parameter measuring the speed of converge to the UFR, UFR represents an ultimate forward rate, t represents time to maturity in the pricing function, N is a number of available par yield rates, u_i are maturities of zero coupon bonds with known prices (or other bonds after conversion into zero bonds) and m_i is market prices of bonds (Smith and Wilson, 2000).

As you can see, there are unknown parameters ξ_i that need to be calculated using a system of linear equations. For each market available bond (and its par yield rate), we can calculate m_i with an final outcome in a form of a vector **m:**

$$m_1 = P(u_1) = e^{-UFR*u_1} + \sum_{j=1}^{N} \xi_j * W(u_1, u_j) \tag{5}$$

$$m_2 = P(u_2) = e^{-UFR*u_2} + \sum_{j=1}^{N} \xi_j * W(u_2, u_j) \tag{6}$$

$$m_N = P(u_N) = e^{-UFR*u_N} + \sum_{j=1}^{N} \xi_j * W(u_N, u_j) \tag{7}$$

Subsequently, we use linear algebra to derive the unknown parameters, the vector **ξ.**

$$\boldsymbol{m} = \mathbf{p} = \boldsymbol{\mu} + \mathbf{W}\boldsymbol{\xi} \tag{8}$$

As we know the vector **μ**:

$$\boldsymbol{\mu} = (e^{-UFR*u_1}, e^{-UFR*u_2}, \dots, e^{-UFR*u_N})^T, \tag{9}$$

we can finally compute the unknown vector **ξ**:

$$\boldsymbol{\xi} = \boldsymbol{W}^{-1}(\boldsymbol{p} - \mu) = \boldsymbol{W}^{-1}(\boldsymbol{m} - \mu). \tag{10}$$

Now, we have all parameters needed to calculate the Smith-Wilson pricing function $P(t)$. Spot rates (and thus the whole spot/zero yield curve) can be then calculated using the following conversion formulas: for continuous compounding $R_t^{zero} = \frac{1}{t} \ln\left(\frac{1}{P(t)}\right)$, for annual compounding $R_t^{zero} = \left(\left(\frac{1}{P(t)}\right)^{\frac{1}{t}}\right) - 1$.

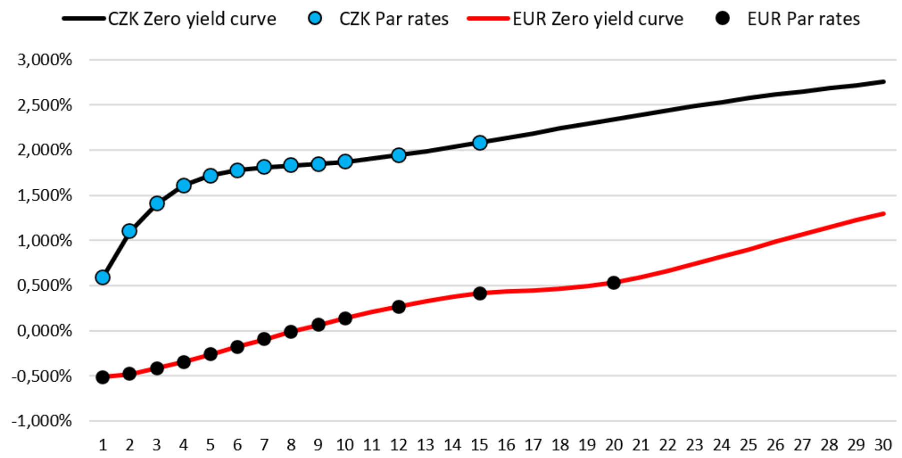
We applied the SW method on 6 country yield curves[1]: CHF is Swiss governmental (risk-free) par curve with 11 liquid tenors; CZK is Czech swap par curve with 10 liquid tenors; EUR is

[1] Data source: Bloomberg database as of 30.4.2021

Euro swap par curve with 11 liquid tenors; GBP is British swap par curve with 12 liquid tenors; JPY is Japanese governmental (risk-free) par curve with 12 liquid tenors; USD is US governmental (risk-free) par curve with 12 liquid tenors.

The charts below include results of the SW method and illustrate the relationships between par rates (dots) and newly created zero yield curves (lines). The analysis shows that both CZK and EUR yield curves did perfectly fit the available par rates (in comparison to the bootstrapping method), see Figure 3.

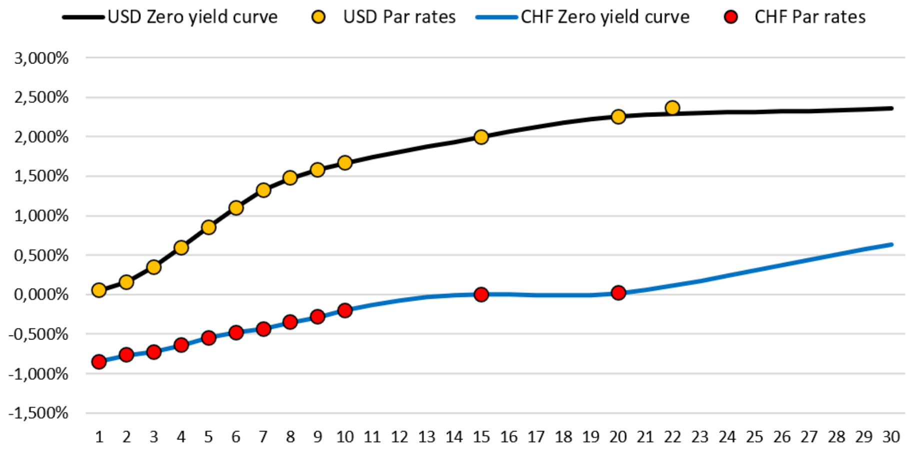
Fig. 3: Yield curves CZK & EUR (30 April 2021)



Source: Bloomberg Terminal data as of 30 April 2021, authors‘ own elaboration

The analysis showed great SW fit for USD and CHF yield curves too. We may see one of the method’s big added values, no need for linear interpolation between available par rates (it was required in the bootstrapping method). SW method calculated the SW pricing function P(t) and this pricing function enables us to derive any tenor (t: time to maturity, in years) which is also beneficial for the long tail of yield curves (tenors > 20 years)

Fig. 4: Yield curves USD & CHF (30 April 2021)



Source: Bloomberg Terminal data as of 30 April 2021, authors' own elaboration

The long tail of yield curves is not only an advantage of the Smith-Wilson method. It is a topic that requires a special attention. The yield curve long tail is dependent on a choice of the UFR (ultimate forward rate). The ultimate forward rate comes out of the hypothesis of expectations which claims that long-term yield rates are an average of short-term yield rates or an average of short-term zero/spot yield rates and short-term forward rates.

The UFR includes investors' expectations regarding the future yield rates of a certain yield curves (eg. CZK or EUR curve). The UFRT represents how much yield an investor may require for an investments that is planned to happen in time T. We can use a parallel to the nominal return on investments that is a sum of a real return and an expected inflation. Applying this approach, we may calculate the ultimate forward rate as

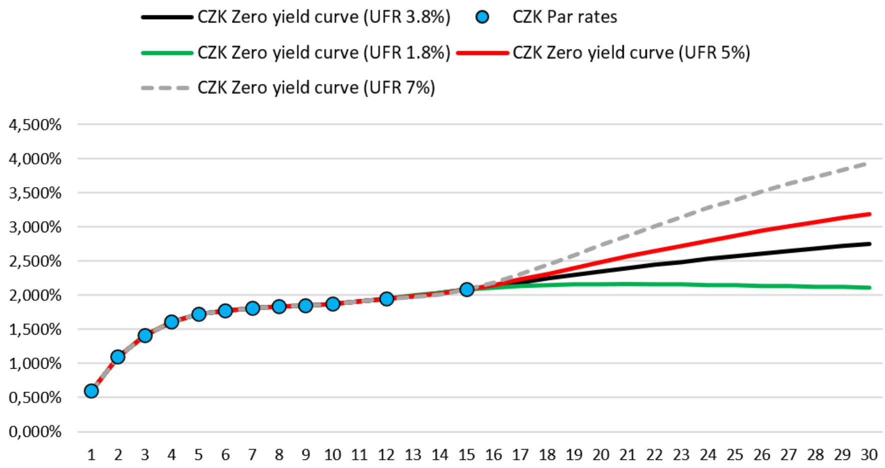
$$UFR_{country} = R_{country} + \pi^{e}_{country}, \tag{11}$$

where

- $\pi^{e}_{country}$ – expected inflation rate in a certain country, inflation target may be used as well
- $R_{country}$ – average yield calculated from liquid par rates (with focus on the middle part of the yield curves, trying to avoid short and long tail extreme rates).

The Czech UFR may be then calculated as 3.8% = 1.8% (average yield) + 2% (inflation target announced by the Czech National Bank). The Czech zero yield curve derived by the Smith-Wilson method with the UFR 3.8% is the blank line below.

Fig. 5: Yield curves CZK with various UFRs (30 April 2021)



Source: Bloomberg Terminal data as of 30 April 2021, authors‘ own elaboration

Grey, red and green lines in chart above illustrate us potential developments of the Czech zero yield curves with various levels of the UFR. The choice of the ultimate forward rate has a material impact on the calculated zero yield curve. Hence, ultimate forward rates need to be selected carefully and be based on solid analytical grounds.

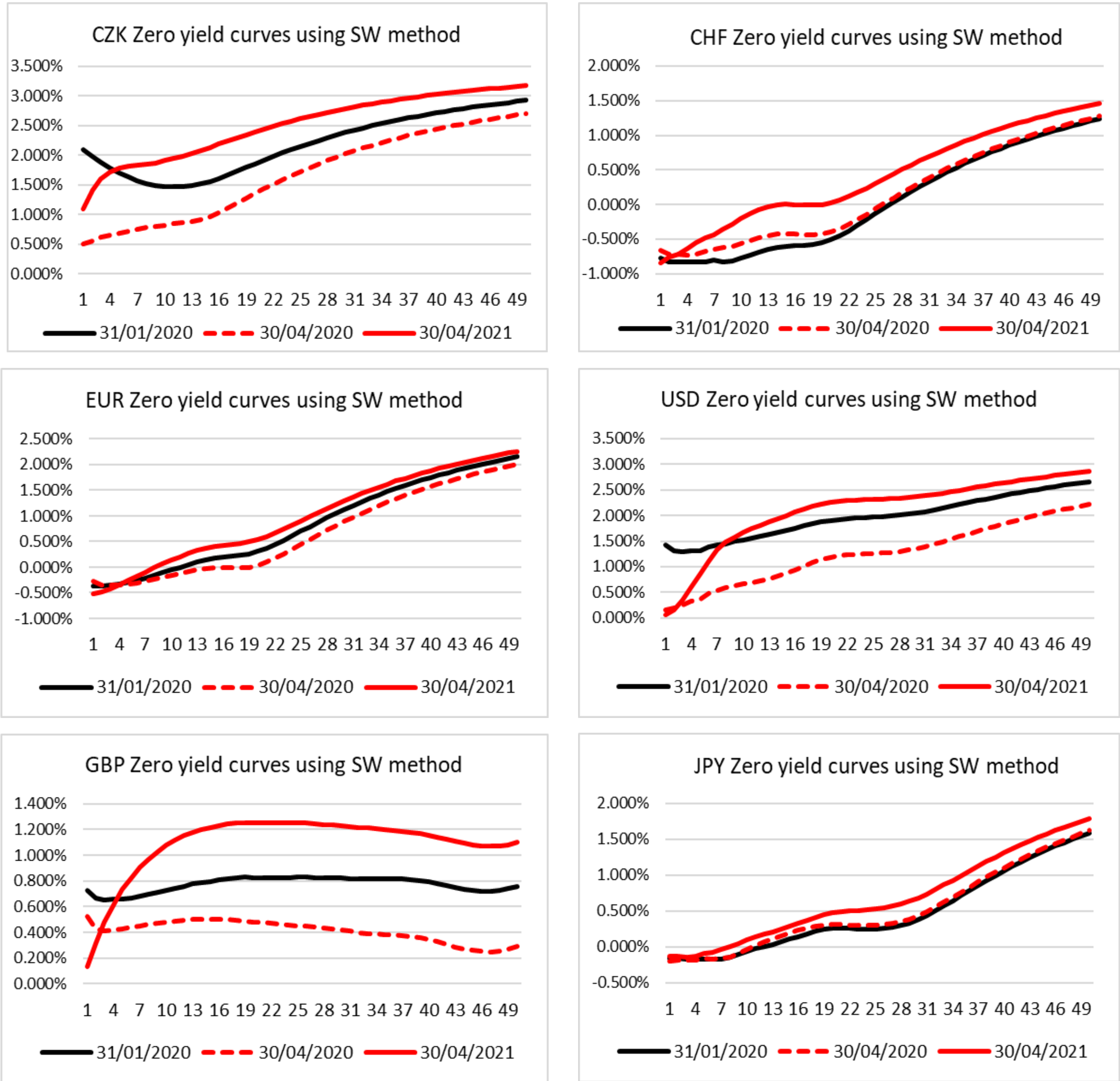
4 Covid-19 impact on yield curves

In the last chapter of the paper, we applied the Smith-Wilson method in the Covid-19 impact assessment. The Covid-19 outbreak in March 2020 influenced strongly whole society including the economy and investors' expectation on yields. Three pandemic relevant reporting dates were thus selected:

- 31.1.2020 (a while before the Covid-19 pandemic outbreak)
- 30.4.2020 (a while after the Covid-19 pandemic outbreak)
- 30.4.2021 (more than one year after the Covid-19 pandemic outbreak)

The Covid-19 pandemic presents a kind of stress period and we tried to test Smith Wilson's method and its stability and reasonability in times of market distress. You can see below results of this stress-test for 6 tested yield curves (country yield curves):

Fig. 6 Results of this stress-test for 6 tested yield curves (country yield curves):



Source: Bloomberg Terminal data as of 30 April 2021, authors' own elaboration

In brief, the pandemic had the strongest impact on the GBP, USD and CZK yield curves. This impact was driven by monetary policies in these countries where central banks directly influenced the short tail of the yield curves by their monetary interventions (eg. they decreased the short interest rates to boost the economic activity that was slowed down by lockdowns). The longer tails are then indirectly influenced by changes in expectations. The impact on the EUR, JPY and CHF zero yield curves was rather smaller. One general observation is an increase in yield curve levels in all 6 countries (with a various magnitude), one year after the start of the pandemic. This increase is caused by an increased risk in all these countries that need to be rewarded by a higher yield. Another observation, based on CZK, EUR, GBP and USD yield curves, is that the short-term reaction to the pandemic was a decrease of the yield curves that was followed by an increase to higher than original levels. This short-term reaction in all 4 markets may be explained by sudden lockdowns and limited amounts of investment opportunities.

Conclusion

The purpose of the paper transformation of par yield rates/curves to zero yield curves. Various types of yield curves were explained. Two method of transformation were presented and tested. The bootstrapping method proved to be a helpful tool for a standardization of various bond (swap) prices and coupons into one single and easy-to-compare zero yield curve. However, the bootstrapping contains some limitations, eg. interpolation and extrapolation issues and accuracy (fit to par rates). Subsequently, the Smith-Wilson method was explained and tested on 6 different yield curves. The method was applied in the stress-test scenario with the Covid-19 pandemic. The results produced by the SW method proved to be reasonable and fit well the par yield rates. The Smith-Wilson method showed consistent results before, during and after the applied stress.

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References

Bloomberg L.P. (2021). Yield curves data for CZK, EUR, CHF, GBP, JPY, USD. 1/1/20 to 1/5/21. Retrieved from Bloomberg database.

Bohn, A., Elkenbracht-Huizing, M., Adler, M. (2014). The Handbook of Alm in Banking: Interest Rates, Liquidity and the Balance Sheet. Risk Books, a Division of Incisive Media Investments.

Choudhry, M. (2019). Analysing and interpreting the yield curve. John Wiley & Sons, p. 359.

Christensen, J. H. E., Diebold, F. X., Rudebusch, G. D. (2011). The affine arbitrage-free class of Nelson–Siegel term structure models. Journal of Econometrics, 164(1), 4–20. https://doi.org/10.1016/j.jeconom.2011.02.011

Christensen, J. H. E., Lopez, J. A., Rudebusch, G. D. (2015). A probability-based stress test of Federal Reserve assets and income. Journal of Monetary Economics, 73, 26–43. https://doi.org/10.1016/j.jmoneco.2015.03.007

Diebold, F. X., Li, C., Yue, V. Z. (2008). Global yield curve dynamics and interactions: A dynamic Nelson–Siegel approach. Journal of Econometrics, 146(2), 351–363. https://doi.org/10.1016/j.jeconom.2008.08.017

Patrick S. Hagan & Graeme West (2006) Interpolation Methods for Curve Construction, Applied Mathematical Finance, 13:2, 89-129. https://doi.org/10.1080/13504860500396032

Rebonato, R. (2015). Yield Curve Modeling and Forecasting—The Dynamic Nelson–Siegel Approach. Quantitative Finance, 15(10), 1609–1612. https://doi.org/10.1080/14697688.2015.1005351

Siegel, A. F., Nelson, C. R. (1988). Long-Term Behavior of Yield Curves. The Journal of Financial and Quantitative Analysis, 23(1), 105–110. https://doi.org/10.2307/2331027

Smith, A., Wilson, T.(2000). Fitting Yield Curves with Long Term Constraints. Parkside House Ashley Road Epsom Surrey, s. 11.

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