NUMERICAL ALGORITHMS FOR STOCHASTIC PROCESSES

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Abstract

In probability theory, a stochastic process is a mathematical object usually defined as a family of random variables. Based on their properties (and their mutual connections), stochastic processes can be divided into various categories, which include random walks, Markov processes, Gaussian processes, Lévy processes, random fields and branching processes.

They are widely used as mathematical models of some systems that appear to vary in a random manner. They are used for modelling in many disciplines such as physics (Papanicolaou, 1995), chemistry (Kampen, 2007), biology (Ricciardi, 1977), climatology (Lions, Temam & Wang, 1992), social sciences (Cobb, 1981) as well as finance (Mayer-Brandis & Proske, 2004) and option pricing (Schoutens, 2003).

The most important example of a stochastic process is the Brownian motion named after Robert Brown who studied the movement of a microscopic particle in water almost two centuries ago. His work was followed by many mathematicians (such as Norbert Wiener or Paul Lévy), who provided mathematical background which later became the foundation of the stochastic analysis, but his work was also followed by many physicists (such as Albert Einstein, Marian Smulchowski or Jean Baptiste Perrin) who studied diffusion of particles suspended in fluid.

Key words: Stochastic process, Brownian motion, simulation

JEL Code: C60, C63

Introduction

The aim of this paper is to introduce some of the most important stochastic processes with focus on the practical implementation of their simulation. As the programming language for our algorithms, we have chosen the program R. It is free software that is used mostly by the statisticians and we believe that the implementation is intuitive and understandable. For more detail, see the brief R manual "An Introduction to R" that comes with every installed version of R.

The main source for this work is the book (Iacus, 2010). There can be found many advanced methods of simulations not only for the basic stochastic processes, but also for more complicated (and rather general) stochastic partial differential equations. The other half of the book is dedicated to the parametric estimation, making it a useful guide to the practical point of view of stochastic processes.

We have used the book (Iacus, 2010) in a way that we summarized the basic notions on the chosen stochastic processes and we improved some of the stated programming codes by removing some errors and by generalization of some of the codes. Hence we created even more concise paper for beginners, learners and students of this topic.

The article is organised as follows. In Section 1, we introduce the standard Brownian motion and present two possible methods of its simulation. The first one comes directly from the basic properties of Brownian motion, the second one is due to Donsker's theorem (see, e.g., (Billingsley, 2013)). We also discuss the space-shifted and time-shifted Brownian motion that is generalized even more in Section 2, where we introduce the Brownian bridge.

Section 3 is dedicated to the geometric Brownian motion. We provide motivation which leads to the definition of the process in the form of stochastic differential equation (1) as well as its solution (4). Two methods of simulation are presented and the influence of the parameters of rate and volatility is discussed. We conclude with Section 4, where we define and simulate trajectories of the Ornstein-Uhlenbeck process.

1 Brownian motion

Brownian motion (also Wiener process) is the fundamental process in the theory of stochastic processes. It is a key process in terms of which more complicated stochastic processes can be described. The most common definition of the Brownian motion $W = \{W(t), t \geq 0\}$, is the characterization by the following properties:

- $W(0) = 0$ with probability 1,
- W has independent increments, i.e., for every $t_n > \ldots > t_1 \geq 0$, the random variables $W_{t_n} - W_{t_{n-1}}, \ldots, W_{t_2} - W_{t_1}$ are independent,
- W has Gaussian increments, i.e., for every $t > s \geq 0$, the increment $W_t - W_s$ is normally distributed with mean 0 and variance $t - s$, $W_t - W_s \sim N(0, t - s)$,
- W has continuous path with probability 1.

The first method of simulation of a trajectory of the Brownian motion is precisely according the above definition. Given a fixed time increment $\Delta t > 0$ and the time interval $[0, T]$, it holds true that

$$W(\Delta t) = W(\Delta t) - W(0) \sim N(0, \Delta t) \sim \sqrt{\Delta t} \cdot N(0,1),$$

and (since the increments are independent), it is also true for any other "following" increment, i.e.,

$$W(t + \Delta t) - W(t) \sim N(0, \Delta t) \sim \sqrt{\Delta t} \cdot N(0,1).$$

Thus we divide the interval $[0, T]$ equidistantly $0 = t_1 < \cdots < t_N = T$ with $t_{i+1} - t_i = \Delta t$, set $i = 1, W(0) = W(t_1) = 0$ and iterate the following algorithm:

1. Generate a random number X from the standard Gaussian distribution.
2. $i := i + 1$.
3. $W(t_i) := W(t_{i-1}) + X \cdot \sqrt{\Delta t}$.
4. If $i < N$, go to step 1.
5. Between any two time points t_i and t_{i+1} interpolate the trajectory linearly.

This algorithm can be implemented in the R language as follows.

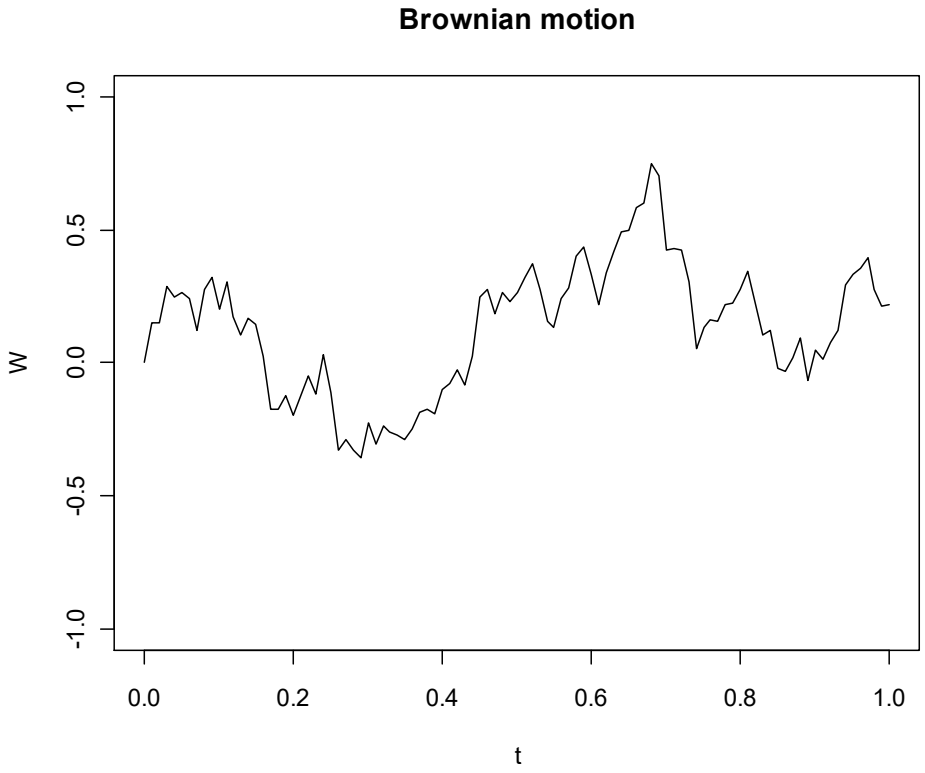
```
> set.seed(222)
> N <- 100 # number of time points
> T <- 1 # length of the interval [0,T]
> Delta <- T/N # time increment
> W <- numeric(N+1) # initialization of the vector W
> t <- seq(0, T, length = N+1) # sequence of time points
> for(i in 2:(N+1))
+ W[i] <- W[i-1] + rnorm(1) * sqrt(Delta)
> plot(t, W, type = "l", main = "Brownian motion", ylim = c(-1,1))
```

While this code is clear and straightforward, the iteration in the `for` cycle is not needed. If we use the function `cumsum` for the cumulative summation, the whole trajectory can be simulated in just one line of R code:

```
> W <- c(0, cumsum(sqrt(Delta) * rnorm(N)))
```

These two implementations are equivalent (they even provide the same trajectory), however due to the nature of R language, the second one is much faster. The simulated path of the Brownian motion can be found in Figure 1.

Fig. 1: A simulated trajectory of the Brownian motion



Source: Author's construction

Brownian motion can be also characterized as the limit of random walks by Donsker's theorem (see (Billingsley, 2013)). It is a functional extension of the central limit theorem:

Let $X_1, X_2, X_3, \ldots$ be a sequence of independent, identically distributed random variables with mean 0 and variance 1, let $S_n = X_1 + \ldots + X_n$ be a partial sum of these variables. Then, as $n \to \infty$,

$$P\left(\frac{S_{[nt]}}{\sqrt{n}} < x\right) \to P(W(t) < x),$$

where $[x]$ is the integer part of the real number x and $t \geq 0$.

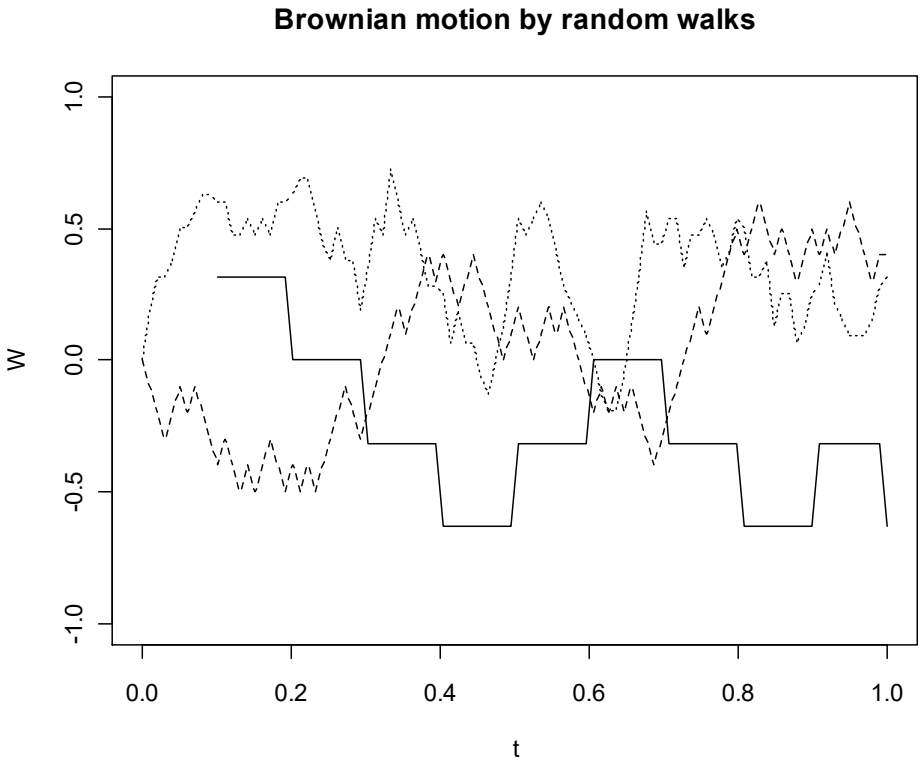
That leads to another way of simulating a trajectory of the Brownian motion: By generating independent random variables $X_1, \ldots, X_n$, which take values 1 and -1 with probability $\frac{1}{2}$, and by their summing and rescaling, we end up with the trajectory of Brownian motion, as is described in the following code:

```
> set.seed(222)
> n <- 10 # number of used variables # also n <- 100 # n <- 1000
> T <- 1
> t <- seq(0, T, length = 100)
```

```
> S <- cumsum(2*(runif(n) > 0.5) - 1)
> W <- sapply(t, function(x) ifelse(n*x > 0, S[n*x], 0))
> W <- as.numeric(W)/sqrt(n)
> plot(t, W, type = "l", main = "Brownian motion by random walks",
  ylim = c(-1, 1))
```

In the above implementation, the random variables X_i, $i = 1, \ldots, n$ were actually generated from the uniform distribution and then they were transformed into ± 1 variables. In Figure 2, we may observe, how many random variables we need to obtain plausible trajectory of Brownian motion.

Fig. 2: Trajectory of the Brownian motion as the limit of random walks – bold line $n = 10$, dashed line $n = 100$, dotted line $n = 1000$



Source: Author's construction

If we are concerning which method to choose, the first method should be used if we are interested in the position of Brownian motion at the fixed time point. (This is quite common in finance in the evaluation of the payoff with a fixed exercise time.) The method is fast and with some small Δt accurate. On the other hand, if we are interested in the whole trajectory of Brownian motion (for example in the evaluation of the Asian option, but also for

the description of the whole dynamic of the process), the second method (with some large number of random variables n, e.g., $n \geq 1000$) could be used.

There are indeed many generalizations of Brownian motion, from which we mention the shifted Brownian motion. Given a Brownian motion $W(t)$ and a point $x \in R$, the process

$$W_{0,x}(t) = x + W(t), \qquad t \geq 0,$$

is a Brownian motion starting from the point x instead of 0 at the time 0.

Furthermore, we may consider not only the shift in the space variable, but also the shift in the time variable. If we want Brownian motion to start at some fixed time t_0 instead of the time 0, we may define

$$W_{t_0,x}(t) = x + W(t) - W(t_0), \qquad t \geq t_0.$$

This is the process $W_{t_0,x} = \{W(t), t_0 \leq t | W(t_0) = x\}$, i.e., it is the standard Brownian motion conditioned in the way that $W(t_0) = x$. Since the increments of Brownian motion are independent on its past (which is called the Markov property, see (Stroock, 2013)), the distribution of $W_{t_0,x}(t)$ and $x + W(t - t_0)$ are equal. Therefore the simulation of this process starts with the simulation of Brownian motion, its shifting in the space (to the point x) and then its translating in the time (so it starts at time t_0). (See the following Section for similar construction.)

2 Brownian bridge

Brownian bridge is an important stochastic process not only in the theory of stochastic processes, but also in some statistical applications. It is a Brownian motion starting at x at time t_0 and terminating at y at time T, $T > t_0$. One of the possible definitions can be as follows

$$W_{t_0,x}^{T,y}(t) = x + W(t - t_0) - \frac{t - t_0}{T - t_0} \cdot (W(T - t_0) - y + x), \qquad T \geq t \geq t_0.$$

This is in fact the process $\{W(t), t_0 \leq t \leq T | W(t_0) = x, W(T) = y\}$, i.e., it is the standard Brownian motion conditioned in the way that $W(t_0) = x$ and $W(T) = y$.

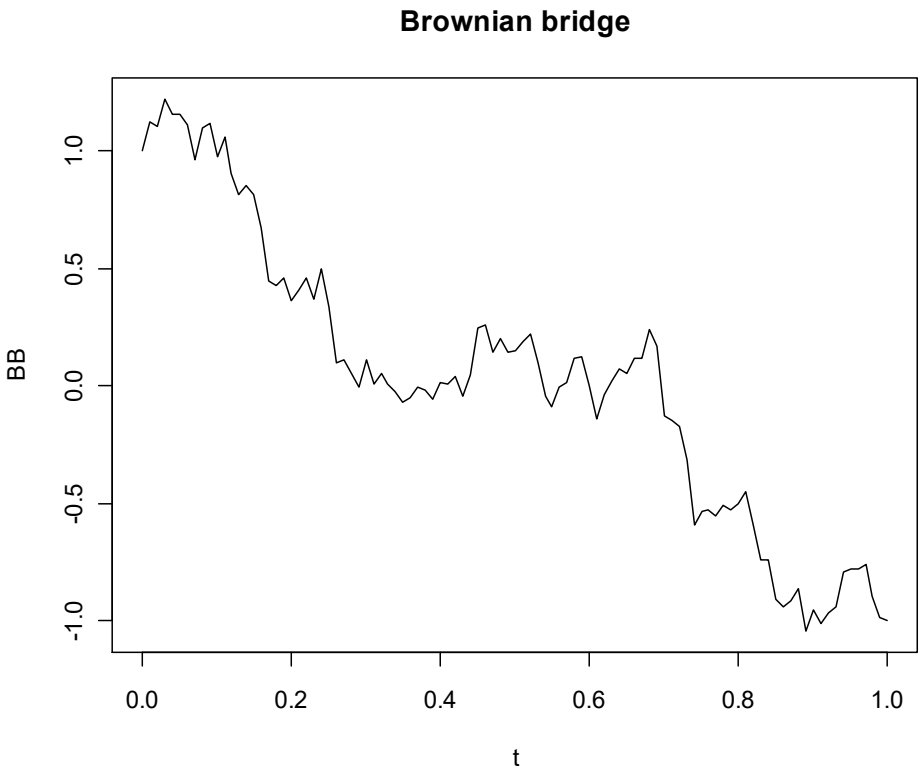
Brownian bridge can be easily simulated by scaling of the trajectory of Brownian motion according the above formula.

```
> set.seed(222)
> N <- 100 # number of time points
> t0 <- 0 # starting time
> T <- 1 # terminal time
> Delta <- (T - t0)/N # time increment
```

```
> W <- numeric(N+1) # initialization of the vector W
> t <- seq(t0, T, length = N+1) # sequence of time points
> W <- c(0, cumsum(sqrt(Delta) * rnorm(N))) # Brownian motion
> x <- 1 # initial value
> y <- -1 # terminal value
> BB <- x + W - (t - t0)/(T - t0) * (W[N+1] - y + x)
> plot(t, BB, type = "l", main = "Brownian bridge")
```

Figure 3 shows simulated trajectory of the Brownian bridge starting from $x = 1$ at time 0 and terminating at $y = -1$ at time $T = 1$.

Fig. 3: A simulated trajectory of the Brownian bridge



Source: Author's construction

3 Geometric Brownian motion

Geometric Brownian motion is used to model stock prices in the Black–Scholes model and it is the most widely used model of stock price behaviour (see (Black & Scholes, 1973)). The process is continuous in time and it is the solution of the following so-called stochastic differential equation

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \qquad S(0) = x, \qquad t \geq 0. \qquad (1)$$

The motivation to the equation (1) is that the variation of the asset $\Delta S = S(t + \Delta t) - S(t)$ in a small interval $[t, t + \Delta t)$ has the following dynamics

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \Delta W. \qquad (2)$$

It means that the returns of the asset consist of some deterministic contribution, which is dependent on the length of the interval Δt (hence the term $\mu \Delta t$) and some stochastic contribution (randomness, noise, shocks, ...), which may behave independently and also complying Gaussian distribution (hence the term $\sigma \Delta W$). Now if we multiply the equation (2) by S and let $\Delta t \to 0$, we arrive at (1).

However, to give a satisfactory meaning to the equation (1), we also have to introduce its integral form

$$S(t) = S(0) + \mu \int_0^t S(u)du + \sigma \int_0^t S(u)dW(u), \qquad t \geq 0. \qquad (3)$$

Since the variation of Brownian motion is not finite and its derivative nowhere exists (see (Karatzas & Shreve, 1998)), there is a need to build a proper stochastic integration with respect to the Brownian motion. It can be done (see also (Karatzas & Shreve, 1998)) and the third term in (3) will obtain a rigorous meaning. The coefficient $\mu > 0$ is interpreted as the interest rate and the coefficient $\sigma > 0$ is interpreted as volatility. For more properties of the geometric Brownian motion, see, e.g., (Ross, 2014).

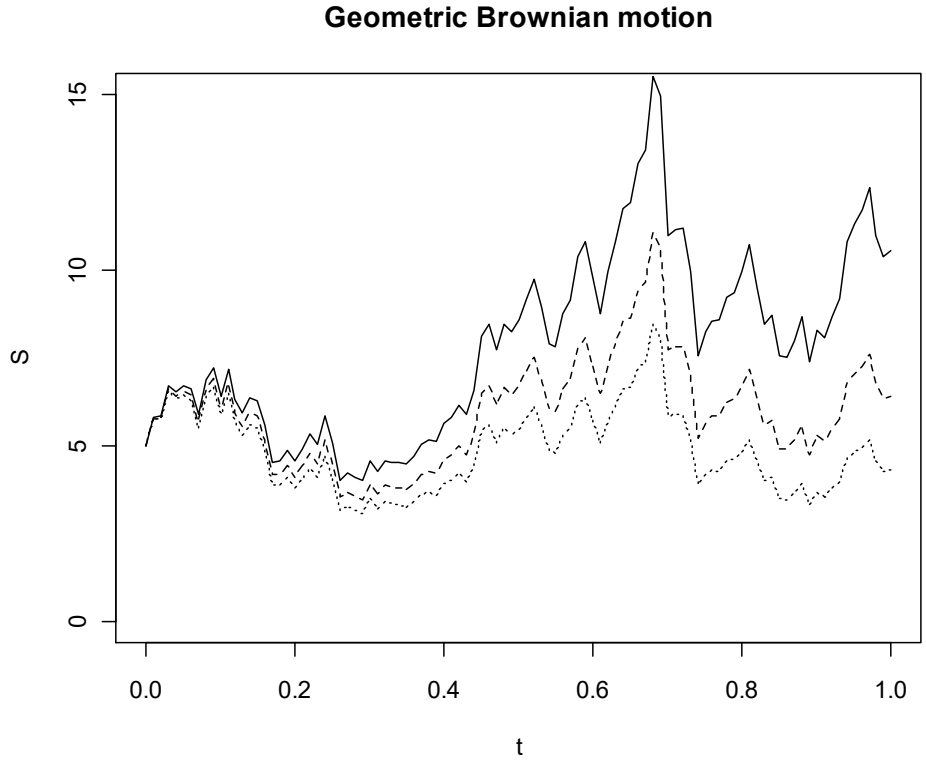
The first way of simulating a trajectory of the geometric Brownian motion comes from the equation (2). We divide the interval $[0, T]$ equidistantly, and (starting from some positive initial value $S(0) = x > 0$) we generate the increments of the process $\{S(t), t \geq 0\}$ according the dynamics (2). This method is called the Euler's method (see (Iacus, 2010)).

```
set.seed(222)
N <- 100 # number of time points
T <- 1 # length of the interval [0,T]
x <- 5 # initial value
mu <- 1 # value of the parameter mu - rate
sigma <- 1 # value of the parameter sigma - volatility
Delta <- T/N # time increment
S <- numeric(N+1) # initialization of the vector S
S[1] <- x
W <- rnorm(N) # generation of increments of Brownian motion
for(i in 1:N)
S[i+1] <- S[i] + mu*S[i]*Delta + sigma*S[i]*sqrt(Delta)*W[i]
```

```
plot(t, S, type = "l", main = "Geometric Brownian motion")
```

Figure 4 depicts three simulated trajectories of the Geometric Brownian motion. In all three cases, the same increments of Brownian motion were used, so we may compare the influence of the parameter μ (rate) on the resulting trajectory.

Fig. 4: A simulated trajectory of the Geometric Brownian motion – bold line $\mu = 1$, dashed line $\mu = 0.5$, dotted line $\mu = 0.1$



Source: Author's construction

There is also another way, how to simulate a path of the geometric Brownian motion. The stochastic differential equation (1) has a unique solution given by the closed formula

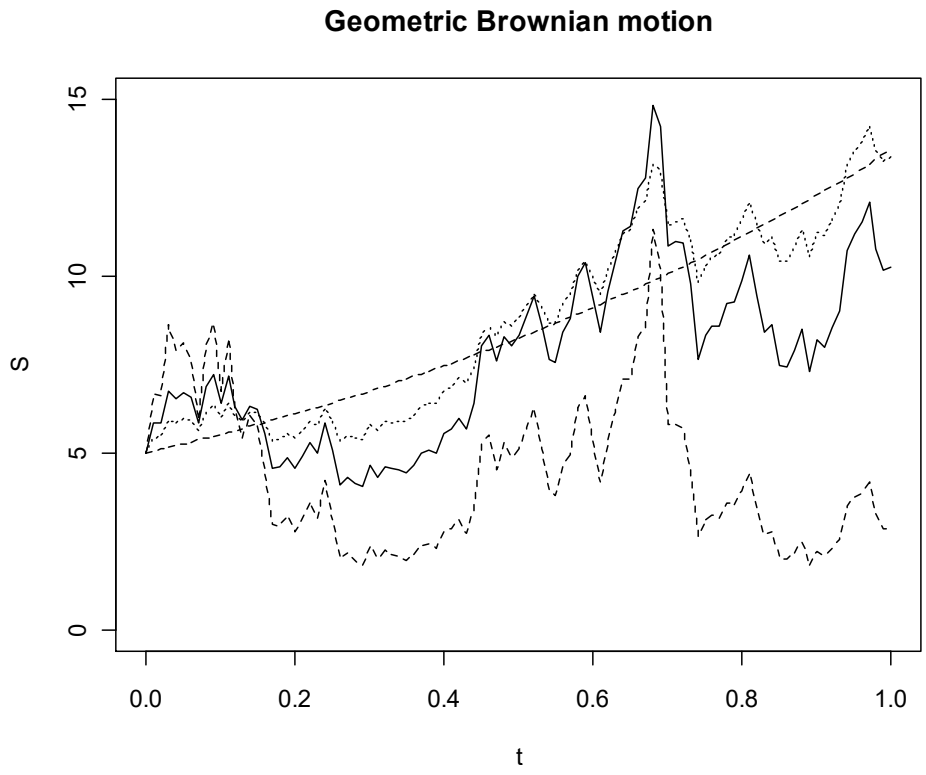
$$S(t) = x \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right\}, \qquad t \geq 0. \qquad (4)$$

The fact that the stochastic process given by (4) has the required stochastic differential (1) can be verified by the Ito's formula (see, e.g. (Iacus, 2010)). With that in mind, the simulation of the trajectory is easy – it is just a function of the standard Brownian motion. See the following implementation and Figure 5.

```
> set.seed(222)
> N <- 100 # number of time points
> T <- 1 # length of the interval [0,T]
```

```
> x <- 5 # initial value
> mu <- 1 # value of the parameter mu - rate
> sigma <- 1 # value of the parameter sigma - volatility
> Delta <- T/N # time increment
> W <- numeric(N+1) # initialization of vector W
> t <- seq(0, T, length = N+1) # sequence of time points
> W <- c(0, cumsum(sqrt(Delta) * rnorm(N))) # Brownian motion
> S <- x*exp((mu - sigma^2/2)*t + sigma*W)
> plot(t, S, type = "l", main = "Geometric Brownian motion")
```

Fig. 5: A simulated trajectory of the Geometric Brownian motion by direct formula – bold line $\sigma = 1$, dashed line $\sigma = 2$, dotted line $\sigma = 0.5$



Source: Author's construction

The generated trajectory with $\mu = 1$ and $\sigma = 1$ (in bold) is actually the same as the trajectory with the same parameters in Figure 4. The two methods actually coincide. Therefore we focused on different values of volatility σ. With larger value of σ, the process $\{S(t), t \geq 0\}$ is further its mean value (which is the function $xe^{\mu t}$ depicted in a smooth dashed line). If we consider the smaller values of σ, the process follows its mean value much closer.

4 Ornstein-Uhlenbeck process

Another stochastic process with applications both in financial mathematics and the physical sciences is the Ornstein-Uhlenbeck process. The process is also continuous in time and it is the solution of the following stochastic differential equation

$$dX(t) = \big(\theta_1 - \theta_2 X(t)\big)\, dt + \theta_3 dW(t), \qquad X(0) = x, \qquad t \geq 0, \qquad (5)$$

where $x \in R$, $\theta_1, \theta_2 \in R$ and $\theta_3 > 0$.

The model with $\theta_1 = 0$ was introduced by Ornstein and Uhlenbeck (see (Uhlenbeck & Ornstein, 1930)) and with the reparametrization

$$dX(t) = \theta\big(\mu - X(t)\big)\, dt + \sigma\, dW(t), \qquad X(0) = x, \qquad t \geq 0,$$

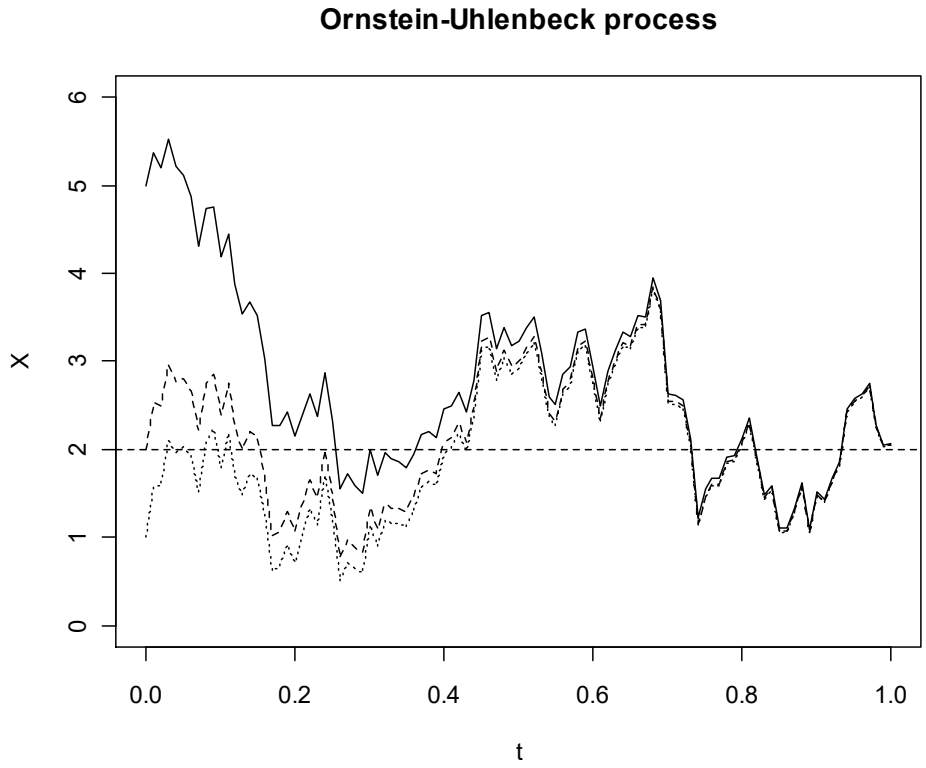
is used in financial mathematics as Vasicek model (Vasicek, 1977). (Here μ is interpreted as the long-run equilibrium of the process, θ as the speed of the reversion and σ as the volatility.)

The simulation of the process follows the definition (5) and the Euler's method is implemented as follows.

```
> set.seed(222)
> N <- 100 # number of time points
> T <- 1 # length of the interval [0,T]
> x <- 5 # initial value # also x <- 2 # x <- 1
> theta <- c(10, 5, 3.5) # values of the parameters
> Delta <- T/N # time increment
> X <- numeric(N+1) # initialization of vector X
> X[1] <- x
> t <- seq(0, T, length = N+1) # sequence of time points
> W <- rnorm(N) # generating of increments of Brownian motion
> for(i in 1:N)
+ X[i+1] <- X[i] + (theta[1] - theta[2]*X[i]) * Delta +
+ theta[3] * sqrt(Delta) * W[i]
> plot(t, X, type = "l", main = "Ornstein-Uhlenbeck process")
```

Figure 6 shows three trajectories of the Ornstein-Uhlenbeck process, which differ in initial value x, but they are all slowly attracted to the equilibrium state $\mu = 2$.

Fig. 6: A simulated trajectory of the Ornstein-Uhlenbeck process – bold line $x = 5$, dashed line $x = 2$, dotted line $x = 1$



Source: Author's construction

However, there also exists a closed formula for the unique solution of equation (5). That is the process

$$X(t) = \frac{\theta_1}{\theta_2} + \left(x - \frac{\theta_1}{\theta_2}\right) e^{-\theta_2 t} + \theta_3 \int_0^t e^{-\theta_2 (t-u)} dW(u), \qquad t \geq 0. \qquad (6)$$

Its stochastic differential $dX(t)$ can also be computed by the Ito's formula and it satisfies (5) (see, e.g., (Iacus, 2010)).

Another way of simulation of the Ornstein-Uhlenbeck process is due to the equation (6). The main advantage is that this is the direct formula, however there is a need to simulate the stochastic integral with respect to the underlying Brownian motion. Therefore we also start with the partition of the interval $[0, T]$ to smaller time steps $0 = t_1 < \cdots < t_N = T$, $t_i - t_{i-1} = \Delta t$, and we approximate the integral by the following sum

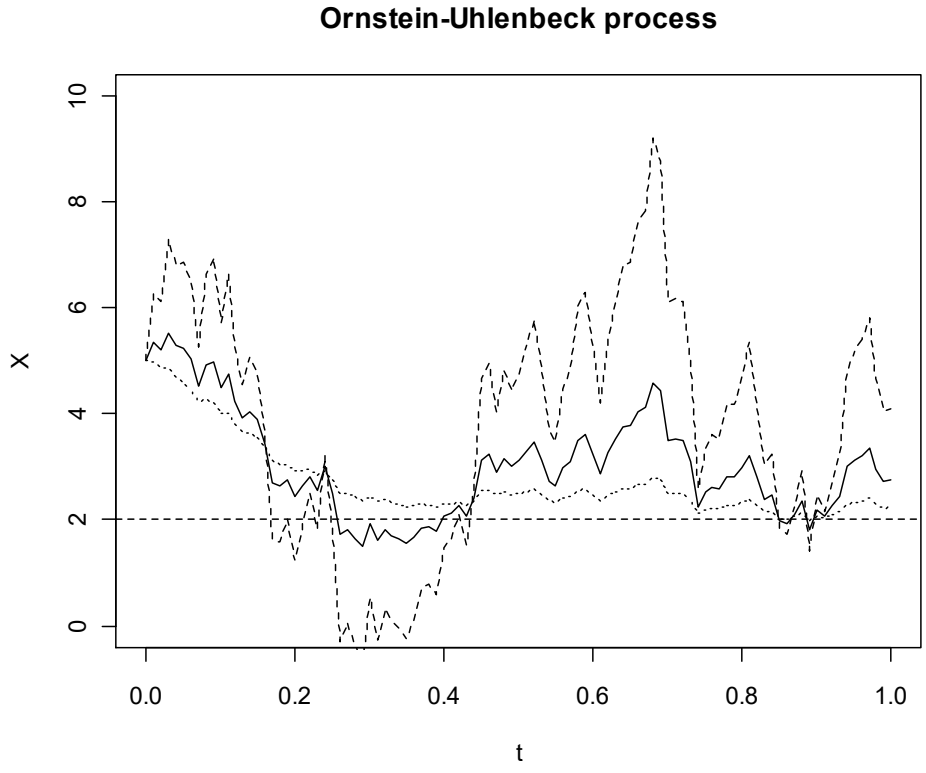
$$\int_0^t e^{-\theta_2 (t-u)} dW(u) \cong \sum_{i=2}^{N} \int_{t_{i-1}}^{t_i} e^{-\theta_2 (t_i - u)} dW(u) \cong \sum_{i=2}^{N} e^{-\theta_2 (t_i - t_{i-1})} \big(W(t_i) - W(t_{i-1})\big).$$

It is indeed a useful way how to handle the stochastic integral generally. Note that the necessary increments of the Brownian motion satisfy $W(t_i) - W(t_{i-1}) \sim N(0, \Delta t) \sim \sqrt{\Delta t} \cdot N(0,1)$ (see Section 1). The following code describes whole simulation.

```
> set.seed(222)
```

```
> N <- 100 # number of time points
> T <- 1 # length of the interval [0,T]
> x <- 5 # initial value
> theta <- c(10, 5, 3.5) # values of the parameters
> Delta <- T/N # time increment
> X <- numeric(N+1) # initialization of vector X
> X[1] <- x
> t <- seq(0, T, length = N+1) # sequence of time points
> stoch.integral <- c(0, sapply(2:(N+1), function (x){
+ exp(-theta[2]*(t[x] - t[x-1]))*sqrt(Delta)*rnorm(1)}))
> X <- sapply(1:(N+1), function(x){
+ theta[1]/theta[2]+(X[1]-theta[1]/theta[2])*exp(-theta[2]*t[x]) +
+ theta[3]*sum(stoch.integral[1:x])})
> plot(t, X, type = "l", main = "Ornstein-Uhlenbeck process")
```

Fig. 7: A simulated trajectory of the Ornstein-Uhlenbeck process by formula (6) – bold line $\theta_3 = 3.5$, dashed line $\theta_3 = 10$, dotted line $\theta_3 = 1$



Source: Author's construction

The bold line in Figure 7 is basically the same trajectory as the bold trajectory in Figure 6 – these two methods again coincide. For now, we have chosen to keep the initial

value the same ($x = 5$) and we simulated three trajectories with different parameters of volatility θ_3. All trajectories are approaching the equilibrium state $\mu = 2$, but, again, the larger volatility, the larger the variance of the process itself.

Conclusion

We introduced the most important stochastic processes in theory of probability and summarized some numerical methods and ideas on their simulation. We believe it is a useful overview, because this topic is not paid too much attention to, and it should be helpful to practitioners, programmers and students.

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