RECURSIVE ESTIMATION OF THE MULTIVARIATE EWMA PROCESS

Radek Hendrych – Tomáš Cipra

Abstract

Recursive estimation methods suitable for univariate GARCH models have been recently studied in the literature. They undoubtedly represent attractive alternatives to the standard non-recursive estimation procedures with many practical applications (especially in the context of high-frequency financial data). It might be truly advantageous to adopt numerically effective techniques that can estimate, monitor, and control such models in real time. The aim of this contribution is to extend this methodology to the multivariate EMWA process by applying general recursive estimation instruments. The multivariate exponentially weighted moving average (MEWMA) model is a particular modelling scheme advocated by RiskMetrics that is capable of predicting the current level of financial time series covolatilities. In particular, the suggested approach seems to be useful for various multivariate financial time series with (conditionally) correlated components. Monte Carlo experiments are performed in order to investigate statistic features of the proposed estimation algorithm. Moreover, an empirical financial analysis demonstrates its capability.

Key words: conditional correlation, covolatility, EWMA, GARCH, recursive estimation

JEL Code: C01, C51, C58

Introduction

Volatility modelling plays the key role in analysis of univariate financial time series. Moreover, understanding of comovements of more financial time series (e.g. various financial asset returns) is also of great practical importance since financial volatilities can move together over time across assets and markets. The models of such covolatilities are relevant tools for better decision-making in portfolio and risk management or asset pricing (see Aielli (2013), Clements, Doolan, Hurn, and Becker (2009), Tse and Tsui (2002)).

The multivariate GARCH (MGARCH) processes (which are commonly used in order to model conditional covolatilities in financial practice) are rarely estimated recursively. However, it might be truly advantageous to adopt recursive estimation algorithms: to evaluate the parameter estimates at a time step, recursive methods operate only with the actual measurements and with the parameters estimated in previous steps. It is in a sharp contrast to the non-recursive (batch) estimation where all data are collected at first and then the given model is fitted. The recursive estimation techniques are effective in terms of memory storage and computational complexity; this efficiency can be employed just in the framework of high-frequency financial time series data. Alternatively, it is possible to apply these methods to monitor or forecast (co)volatilities, to evaluate risk measures, to detect faults, or to check model stability including detection of structural changes in real time. (see e.g. Aknouche and Guerbyenne (2006), Dalhaus and Rao (2007), Cipra and Hendrych (2018), Hendrych and Cipra (2018)). On the other hand, one should respect that the multivariate GARCH models are relatively complex; one must identify and calibrate large dynamic systems with many parameters under specific constraints by means of data that are covolatile in time. Therefore, we confine ourselves in this paper to the simplest MGARCH model, i.e. the multivariate EWMA process with only unknown parameter to be estimated.

The aim of this contribution is to introduce a recursive estimation scheme for the multivariate EMWA process by applying the general recursive prediction error method. The paper is organized as follows: Section 1 presents the multivariate EWMA process. Section 2 proposes a recursive estimation algorithm suitable for the multivariate EMWA model. Section 3 demonstrates the convergence behavior of the suggested estimation method. Section 4 investigates the currency data by employing the introduced estimator.

1 Multivariate EWMA process

Let us consider a multivariate stochastic vector process $\{\boldsymbol{r}_t\}_{t\in\mathbb{Z}}$ of dimension $(m\times 1)$. Denote Ω_t the σ-algebra generated by observed time series up to and including time t, i.e. $\Omega_t = \sigma(\boldsymbol{r}_s;\, s \le t)$ is the smallest σ-algebra with respect to which $\boldsymbol{r}_s$ is measurable for all $s \le t$, s, $t \in \mathbb{Z}$. In this framework, assume the following general model of conditional covolatilities (i.e. conditional covariance matrices) when ignoring a potential nonzero conditional mean vector:

$$\boldsymbol{r}_t = \mathbf{H}_t^{1/2} \cdot \boldsymbol{\varepsilon}_t\,, \tag{1}$$

where $\boldsymbol{\varepsilon}_t$ are *iid* random vectors with zero mean and identity covariance matrix, $\mathbf{H}_t$ is an $(m\times m)$ symmetric positive definite Ω_{t-1}-measurable matrix with the square root matrix denoted as $\mathbf{H}_t^{1/2}$ such that $\mathbf{H}_t = \mathbf{H}_t^{1/2}(\mathbf{H}_t^{1/2})^{\mathrm{T}}$. The matrix $\mathbf{H}_t$ represents the covariance matrix given Ω_{t-1}. In fact, the conditional moments of $\{\boldsymbol{r}_t\}$ are obviously:

$$\mathrm{E}(\boldsymbol{r}_t|\Omega_{t-1}) = \mathbf{0}, \quad \mathrm{cov}(\boldsymbol{r}_t|\Omega_{t-1}) = \mathbf{H}_t .$$

The additional part of the general modelling scheme (1) is the so-called covolatility equation for the conditional covariance matrix $\mathbf{H}_t$. It determines the type of the corresponding model and contains unknown parameters ordered in a column vector $\boldsymbol{\theta}$ (therefore, one should write $\mathbf{H}_t(\boldsymbol{\theta})$ or even $\mathbf{H}_{t\,|\,t-1}(\boldsymbol{\theta})$ to be more exact). Point out that all vectors without transposition signs in this text are supposed to be column wise.

The multivariate exponentially weighted moving average model (MEWMA) is a particular modelling scheme advocated by RiskMetrics that is capable of predicting the current level of financial time series covolatilities. In this case, the covolatility equation for the conditional covariance matrix $\mathbf{H}_t$ has the following truly simple form:

$$\mathbf{H}_t = (1-\lambda)\boldsymbol{r}_t\boldsymbol{r}_t^{\mathrm{T}} + \lambda\mathbf{H}_{t-1}, \;\; \lambda \in (0,1) , \tag{2}$$

where λ is the only parameter to be estimated. It means that the method is very parsimonious in parameters (and also constraints on λ are very simple).

2 Recursive estimation of the multivariate EMWA process

Hendrych and Cipra (2018) developed a recursive estimation technique for univariate GARCH models that is effective in terms of memory storage and computational complexity since the current parameter estimates are evaluated using the previous estimates and actual measurements. It can be employed in order to estimate, monitor, and control such models in real time. The algorithm consists in a generalization of recursive prediction error method and can be extended to the multivariate case to construct an estimator of unknown parameters ordered in the column vector $\boldsymbol{\theta}$ by minimizing the loss function (based on the negative conditional quasi log-likelihood criterion):

$$\min_{\boldsymbol{\theta}} \sum_{t=1}^{T} \left[\ln\left|\mathbf{H}_t(\boldsymbol{\theta})\right| + \boldsymbol{r}_t^{\mathrm{T}}\mathbf{H}_t(\boldsymbol{\theta})^{-1}\boldsymbol{r}_t \right]. \tag{3}$$

The corresponding estimation procedure suitable for the MEWMA process is described algorithmically by the system of the following recursive formulas:

$$\hat{\lambda}_t = \hat{\lambda}_{t-1} - \eta_t R_t^{-1}\left[\mathrm{tr}\left(\mathbf{H}_t^{-1}(\hat{\lambda}_{t-1}) \frac{\partial \mathbf{H}_t(\hat{\lambda}_{t-1})}{\partial \lambda} \right) - \boldsymbol{r}_t^{\mathrm{T}}\mathbf{H}_t^{-1}(\hat{\lambda}_{t-1}) \frac{\partial \mathbf{H}_t(\hat{\lambda}_{t-1})}{\partial \lambda} \mathbf{H}_t^{-1}(\hat{\lambda}_{t-1})\boldsymbol{r}_t \right], \tag{4}$$

$$R_t = R_{t-1} + \eta_t \left[\operatorname{tr}\left(\mathbf{H}_t^{-1}(\hat{\lambda}_{t-1}) \frac{\partial \mathbf{H}_t(\hat{\lambda}_{t-1})}{\partial \lambda} \mathbf{H}_t^{-1}(\hat{\lambda}_{t-1}) \frac{\partial \mathbf{H}_t(\hat{\lambda}_{t-1})}{\partial \lambda} \right) - R_{t-1} \right], \tag{5}$$

$$\mathbf{H}_{t+1}(\hat{\lambda}_t) = (1 - \hat{\lambda}_t) \boldsymbol{r}_t \boldsymbol{r}_t^{\mathsf{T}} + \hat{\lambda}_t \mathbf{H}_t(\hat{\lambda}_{t-1}), \tag{6}$$

$$\frac{\partial \mathbf{H}_{t+1}(\hat{\lambda}_t)}{\partial \lambda} = -\boldsymbol{r}_t \boldsymbol{r}_t^{\mathsf{T}} + \mathbf{H}_t(\hat{\lambda}_{t-1}) + \hat{\lambda}_t \frac{\partial \mathbf{H}_t(\hat{\lambda}_{t-1})}{\partial \lambda}, \tag{7}$$

$$\eta_t = \frac{1}{1 + \xi_t / \eta_{t-1}} \text{ for } \xi_t = \tilde{\xi} \cdot \xi_{t-1} + (1 - \tilde{\xi}),\ \xi_0, \tilde{\xi} \in (0,1), \eta_0 = 1. \tag{8}$$

The application of forgetting factor ξ_t is typical in literature on the identification of dynamic systems since it improves the convergence properties including the statistical consistency of corresponding recursive estimators of the type (4)-(8). For instance, one can use a constant forgetting factor $\xi_t \equiv 0.99$ or an increasing forgetting factor $\xi_t = 0.99\xi_{t-1} + 0.01$, $\xi_0 = 0.95$. The first case is associated with the eventuality that the given parameters can vary over time: the constant forgetting factor less than one progressively reduces the influence of historical measurements, and thus enables to detect parameter changes. The second option corresponds to recursive estimation supposing that the given parameter is time-invariant: the increasing forgetting factor significantly improves the convergence speed of the algorithm during the transient phase (the reason is that early information involved in several first measurements is somewhat misused due to obvious initial uncertainties, and should therefore carry a lower weight when comparing to later measurements, which are processed in a better way, see Ljung and Söderström (1983, Chapter 5)). The algorithm initialization described by Hendrych (2015) might be straightforwardly adapted for the herein proposed technique. Furthermore, at each time t it is necessary to check whether the recursive estimate belongs to the interval (0,1) before evaluating other quantities in (4)-(8). If not, one should artificially set the current estimate as the previous one to avoid eventual specification problems. This simple projection ensures positivity of the conditional covariance.

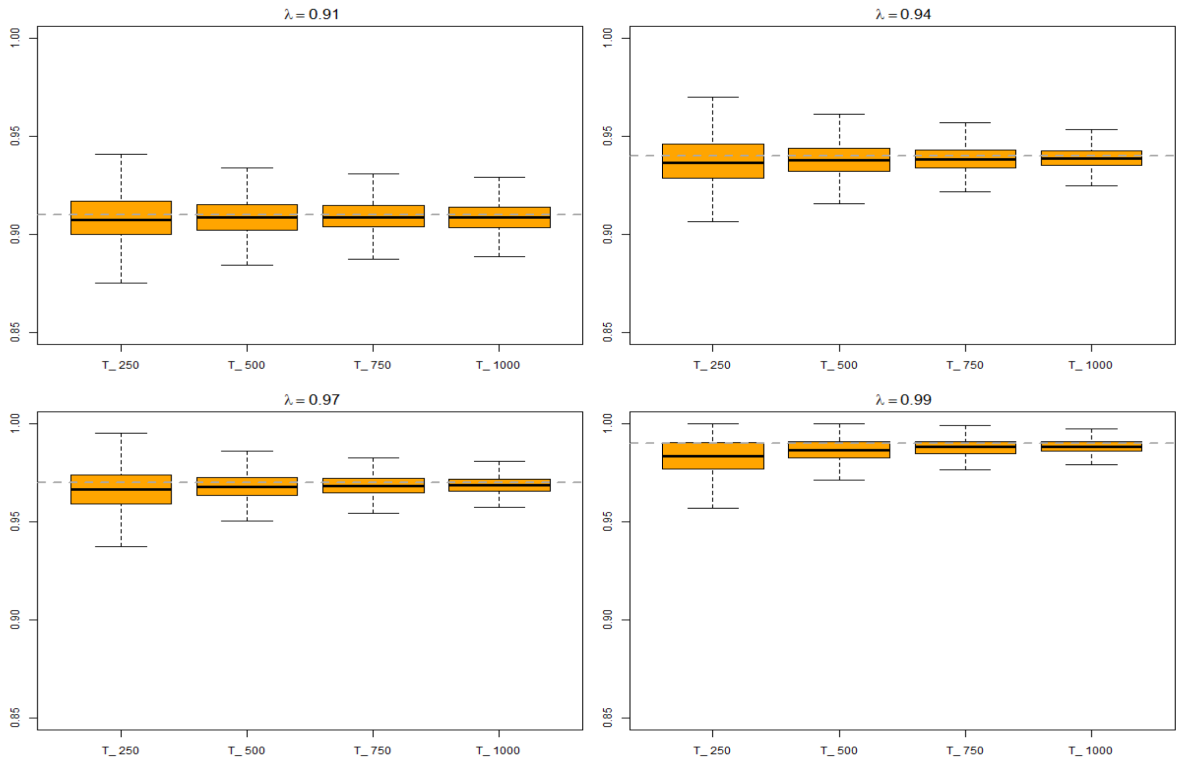
The theoretical properties of the suggested recursive estimation algorithm coincide with the conventional non-recursive case (as t goes to infinity), when the corresponding negative conditional log-likelihood criterion is minimized. Namely, convergence and asymptotic distributional properties are identical for a sufficiently large number of observations. Refer to Ljung and Söderström (1983, Chapter 4) for the theoretical features of the prediction error method; their derivation employs instruments from the ordinary differential equation theory.

3 Monte Carlo experiments

This Section briefly investigates the proposed recursive estimation procedure (4)-(8) by means of Monte Carlo simulations. Various simulation experiments have been performed with almost analogical conclusions for various parametric configurations. Therefore, only selected bivariate representative cases are reviewed here. In particular, the bivariate MEWMA process of the length 1000 was simulated using $\lambda = 0.91$, 0.94, 0.97, and 0.99 (refer to (2)) and applying standardized bivariate Gaussian innovations (with one thousand repetitions).

Figure 1 displays time records of the corresponding boxplots which represent the distribution of recursive estimates in times $t = 250$, 500, 750, and 1000. Evidently, the suggested recursive estimates converge to the true values of parameters λ jointly with decreasing variances. Thus, one might conclude that the suggested recursive method (4)-(8) is capable of estimating the only MEWMA parameter in accordance with the literature (Ljung, 1999). Note that the forgetting factor ξ_t was specified as $\xi_t = 0.99\xi_{t-1} + 0.01$, $\xi_0 = 0.95$.

Fig. 1: Simulation results for recursive estimation of the MEWMA process



Source: Simulations performed by authors using statistical software **R**

4 Case study

The main goal of this Section is to investigate conditional covolatilities (correlations) among selected currency pairs employing the recursively estimated multivariate EMWA. Particularly, we shall discuss the dynamic interconnection of the Czech crown (CZK) with other currencies. The estimated covolatilities (correlations) of corresponding currency pairs are indicators of the mutual currency risk that plays an important role in various regulatory systems in banking and insurance or for constructing statistical prognosis in economy.

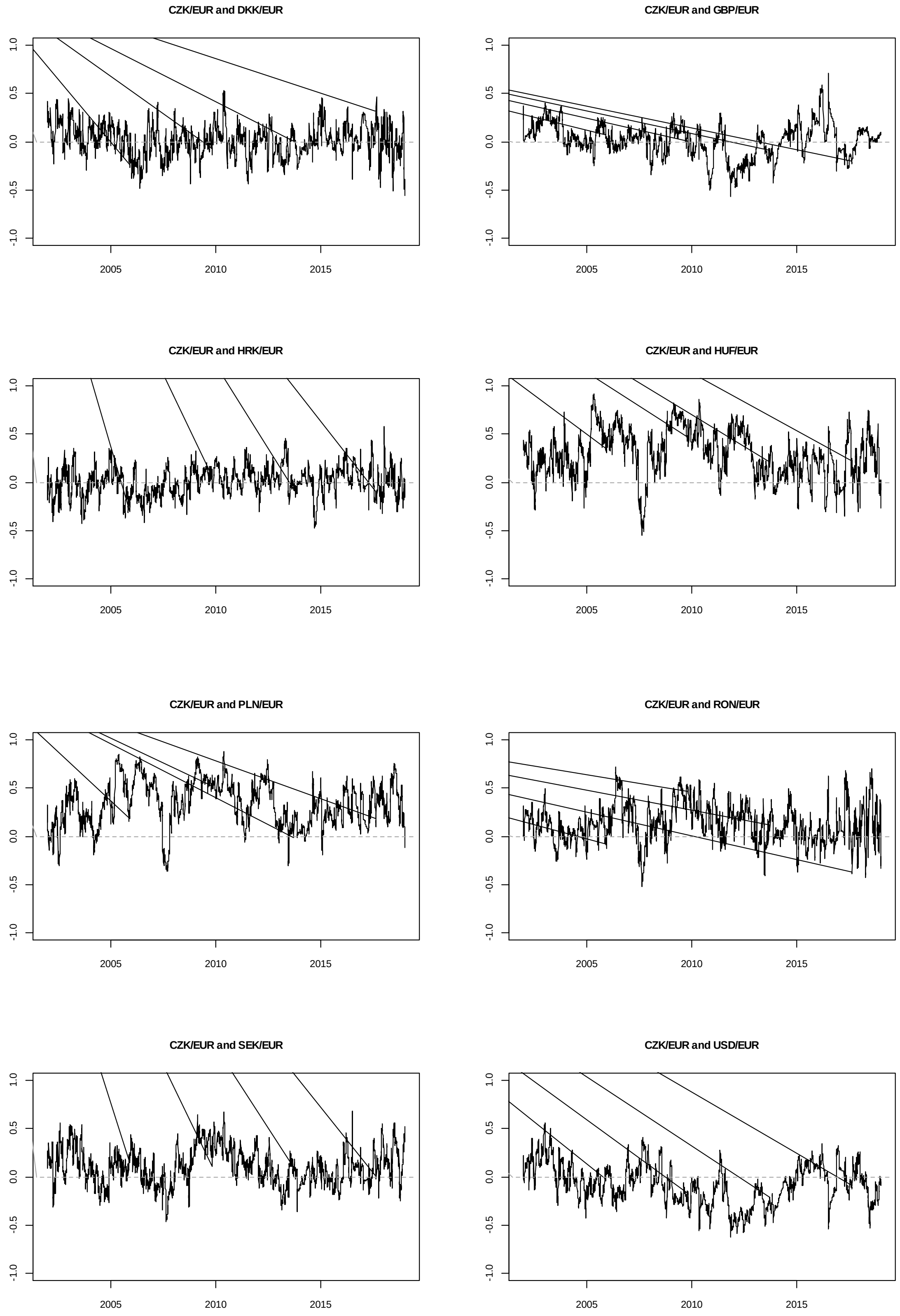
More explicitly, we analyze bivariate conditional correlation (or covariance) matrices between the log-returns of daily currency rate CZK/EUR against other eight currency rates (all denominated by EUR), namely: Croatian kuna (HRK), Danish krone (DKK), Hungarian forint (HUF), Polish zloty (PLN), Romanian leu (RON), Swedish krona (SEK), UK pound sterling (GBP), US dollar (USD). The time range of the study is Jan 2001-Dec 2018 according to ECB[1], i.e. one deals with eight bivariate processes $\{\boldsymbol{r}_t\}$ of length $T = 4605$ observations. Each of them has the same first component, namely the log-returns of daily currency rate CZK/EUR.

For each of eight pairs of log-returns of daily currency rates $\boldsymbol{r}_t = (r_{1,t}, r_{2,t})^{\mathrm{T}}$, one has recursively estimated the bivariate MEWMA model. We have employed the recursive estimation algorithm (4)-(8) respecting various technicalities involved when applying this procedure for real data (e.g. the forgetting factor $\xi_t = 0.99\xi_{t-1} + 0.01$, $\xi_0 = 0.95$, was used).

Figure 2 presents conditional correlations among the daily log-returns of exchange rates CZK/EUR and */EUR for eight studied currencies recursively estimated by means of the MEWMA model in period Jan 2001-Dec 2018. One identifies that the used model approach injects an important dynamic aspect to the analysis of mutual behavior of currencies (sometimes with very intensive covolatilities). Some exchange rates show significant correlation links to the Czech crown. In particular, one notices the relatively high positive correlations between the CZK/EUR against the HUF/EUR or the PLN/EUR. On the contrary, the CZK/EUR is nearly neutral against the HRK/EUR or the DKK/EUR (if we ignore the presence of noise). There is another interesting aspect, namely the conditional correlations can show some trends (refer to the increasing trend of CZK/EUR against the GBP/EUR in the period 2013-2016).

[1] http://sdw.ecb.europa.eu/browse.do?node= 9691296, last access 10th January 2019

Fig. 2: Conditional correlations between daily log-returns of exchange rates CZK/EUR and */EUR recursively estimated using the MEWMA model (Jan 2001-Dec 2018)



Source: Calculation performed by authors using statistical software **R**

The particular models are verified by means of the bivariate Ljung-Box test (Lütkepohl, 2005, p. 171). It is applied in two versions: (*i*) the first version (see the test statistics Q in Table 1) based on estimated bivariate residuals verifies the serial uncorrelatedness and (*ii*) the second version (see test statistics $Q2$ in Table 1) based on squares of estimated bivariate residuals verifies the elimination of heteroscedasticity. The maximum delays of corresponding statistics Q are chosen as the natural logarithm of time series lengths as it is usually recommended.

Table 1 presents p-values for both versions of this test. The obtained results are promising. The (squared) residual uncorrelatedness is rejected on the 5% significance level only in one case for Q and in no case for $Q2$, respectively.

Tab. 1: Multivariate Ljung-Box tests (columns Q and $Q2$ contain p-values)

Exchange rate 1	Exchange rate 2	Q	$Q2$
CZK/EUR	DKK/EUR	0.36637	0.21492
CZK/EUR	GBP/EUR	0.00825	0.57121
CZK/EUR	HRK/EUR	0.36117	1.00000
CZK/EUR	HUF/EUR	0.22317	0.95423
CZK/EUR	PLN/EUR	0.16215	0.08995
CZK/EUR	RON/EUR	0.22871	0.76751
CZK/EUR	SEK/EUR	0.54316	0.37302
CZK/EUR	USD/EUR	0.49079	0.99968

Source: Calculation performed by authors using statistical software **R**

Conclusion

This paper introduced the recursive estimation strategy for the multivariate EWMA process. Monte Carlo simulations demonstrated the convergence behavior of the proposed estimation technique. The realized empirical case study indicated that the application of recursively identified multivariate EWMA model represents an approach that can be useful when one models and controls on-line portfolios of assets with a higher frequency of records and/or potential conditional correlations among components changing in time. The study confirmed various dynamical links among some currencies and outlined potential applications in portfolio management.

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