PREHISTORY OF THE INFINITESIMAL CALCULUS

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Abstract

The discovery of infinitesimal calculus (one of the most important products of human spirit of all times) is usually attributed to English mathematician and physicist Newton and to German mathematician and philosopher Leibniz. However, the problems of measure of change and of limits had been approached and pondered upon by a range of scholars long before them in the past. This article deals with the developments of this issue from the ancient Greek scholar Archimedes who (although ancient Greeks were scared stiff of infinity) probably lay the foundations of integral calculus by the exhaustion method. In Middle Ages, the considerations of infinitely small quantities caught the eyes of Nemorarius, Bradwardin, and Oresme. At the turn of the 16th and 17th centuries the considerations of infinitesimal calculus were on agenda of Kepler and Galilei. And then there followed the era of Newton and Leibnitz who realized that differential calculus and integral calculus are mutually opposing procedures.

Key words: infinitesimal calculus, Archimedes, Oresme, Leibniz, Newton

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Introduction

Infinitesimal calculus, shortly just calculus, is a field of mathematics very close to mathematical analysis. Its main parts are differential calculus and integral calculus with important concepts of "derivation" and "integral". The designation is etymologically based on the term "infinitesimal value", which denotes an infinitesimal or infinitely small number, the absolute value of which is less than any positive real number. More precisely: the number x is infinitesimal if and only if, for any integer n, the following is true: $|nx| < 1$, regardless of the size of n. Infinitesimal calculus is one of the most important discoveries of all times and, although the criteria of aesthetics tend to be more subjective, it can also be seen as one of the most beautiful creations of human spirit. Derivatives and integrals live their separate lives. Appearing and disappearing, approaching one another only to move away from each other, or even get lost forever. They have been among us for a long time. To integrate is actually to measure. When wine was drunk from cups, when gold was transported in tubs, when cloth was ripped to make clothing – in all those instances there was integration.

We will, however, not go that far. Let us start in times closer to our own, in 4th century BC. The ancient Greeks were scared stiff of infinity (Coufal & Tobíšek, 2017), as can be demonstrated by Zeno of Elea (ca. 490 BC – ca. 430 BC) which, according to Aristotle, used dialectics to decompose and defocus the arguments of opponents so that they seemed dubious and ambiguous. Two arguments in particular are renowned, contained in the Zeno paradoxes. Here is one of them:

Achilles and the tortoise – Achilles is in point A, the tortoise is in the lead in point B. Achilles is faster, but when he reaches point B, the tortoise is already in point C. When Achilles arrives at point C, the tortoise is already in point D. The distances between the two points are continuously decreasing, Achilles, however (according to Zeno), can never reach the tortoise.

As the argument makes evident, Zeno radically prefers formal thought to sensual perception. The paradox is reproduced by Aristotle (1910-1952, vol. II, V 239b) in his *Physics*, showing its flaws. The flaw of the reasoning is in the fact that the sum of an infinite series can be finite if its terms decrease sufficiently quickly. And that is the case here. Zeno's consideration is nevertheless one of the first examples of deliberation that lead to the discovery of infinitesimal calculus.

1 Antiquity

In 216 BC Hannibal slashed some 50 000 brave legionnaires at Cannae. A turning point in Roman history – one of those turning points where victory or demise are not decided by physical, but only by mental powers. For Rome, it seemed, everything was lost. But still: the last men able to wield weapons are collected, armed with old weapons from temple loots. Two years later, the consul Marcus Claudius Marcellus (268 BC – 208 BC) and what remains of the army, nearly destroyed at Cannae, stand before the gates of Syracuse in Sicily to punish the Carthaginian allies. The Romans are haunted by peril, and an ill-fated miracle also occurs here. When Marcellus attacks Syracuse from the sea, iron hands and beaks descend from the fortification walls, biting into the ships and rising them up in the air, only to let them fall again. And the splintered wooden boards, behung by the drowning men, are sprayed from above by hailstorm of immense rocks, never before caused by human hand. The oldest of veterans turn dim. As soon as a string of rope or a piece of wood appears above the walls, legionnaires flee in unstoppable horror. They wanted to fight men, or (if need be) Hannibal's war elephants, not hostile gods and two-score-armed giants. Their commander, consul Marcellus, struggles to find an explanation of this horrible miracle (Riečan, 1988). And he learns that they fight a single man, lonesome and old, called Archimedes, aged 72 (ca. 287 BC – 212 Syracuse); the greatest of all Greek mathematicians, one of those weird, world-ignoring men, as little understood by the hard-boiled Romans anchored in reality as their dealings with lines and letters, so obscure to Roman eyes. Is it magic? Have they ridiculed these fools for no good reason? No time for fun now, as magic claws reach from behind the walls and as rocks keep falling as if both Vesuvius and Etna jointly eructed the bowels of Earth. Hiero II, the ruler of the besieged Syracuse, received demands from the Roman commander, also including the handover of two hostages: Hiero's daughter Hellen, and the miraculous scholar Archimedes. His conscience is not entirely clear – he is coming for advice and yet has already responded to Marcellus' demands. Hiero would even let the pretty Hellen go, but was unrelenting in case of Archimedes. Archimedes knew that the exhausted city cannot defend against the Roman numbers infinitely, and turned to his mathematics. When in deep contemplation, he ignored his surroundings, hearing and seeing nothing save for his calculations and drafts. This Archimedes, bred by the Greeks, is supposedly the most ridiculous of all geometers. Romans learn all this from the captive Syracusans who want to use their garrulity to buy easement of their captive fate. Once, when taking bath, he drew lines into sand, murmuring incomprehensible words while being rubbed with oil. Allegedly, once he even ran naked across Syracuse, yelling incessantly one single word: "*Heureka*!" What did he "find"? He found out that the goldsmith cheated on king Hiero II and did not make a golden wreath out of pure gold. He reportedly realized that when his bath overflowed as soon as he entered it. That, by all gods, is no great discovery. This Archimedes must definitely be a fool, or a demon, or both. When, finally and after two distressful years, Syracuse fell in their hands by deceit, they rushed through the streets of the conquered city to murder and plunder, even more ferocious than ever before as they feared a new Archimedean monster could be just around the next corner. In this rush, one legionary entered a house, seemingly abandoned. An old man was sitting in the atrium, drawing diagrams into sand. Why wouldn't he? There is a lot of noise in the city today, but such noise has been no rarity in the past two years. The problem cannot be postponed. "*Noli tangere circulus meos*" (Do not disturb my circles), he uttered, admonishing mildly the Roman legionary, but the sword of the latter at the very same moment of time cut the life of Archimedes to an end. Did the soldier know he killed Archimedes? Did he want to eliminate "the magician" to save the legions and Rome, even though Marcellus ordered them to spare the life Archimedes? Marcellus was furious when he heard of this act. He had Archimedes buried with all honors and built a memorial in his name. It is clear that Archimedes knew a lot about parables and tried to calculate various (surface) areas via "exhaustion". Maybe Archimedes said "*parabolas meas*" rather than "*circulos meos*". Who knows? For Archimedes, nothing was more important at that moment than his circles. He had neither the city on his mind, nor the catapults, nor the threat to his own life. He wanted to find the truth. He felt that the rule on rays passing through the focal point of a parable (differential calculus) and the rule on quadrature of (the area of) a parable (integral calculus) are closely connected – he was, after all, the grand master of the exhaustion method. In 212 BC an old man, aged 74, was killed by the blind ruthlessness of a Roman legionary. It shows that Greek mathematics, in their proud seclusion, only turned to reality when it was too late. The circles of Archimedes were literally blown away by the wind, as they were only drawn in the sand. Maybe they contained the discovery of calculus, a discovery it took eighteen more centuries for humanity to make; only then the Faustian spirit of western nations took up where the Roman soldier, in his blind rage, had destroyed the Titan circles (Coufal & Tobíšek, 2017; Cantor, 1880; Riečan, 1988).

2 Middle Ages and the Threshold of Modernity

Greek mathematics just like its protective goddess Pallas Athena sprang fully armed from the head of Zeus, maintaining itself as a protected art. It grew stronger and bigger, but it gradually lost relation to life, stagnating and dying. Similar phenomena can be observed both in the classical period of mathematics and at the end of the Middle Ages and in Modernity. And, of course, there are also various internal and external influences in Europe, first with the overarching and connecting influence of the Roman Catholic Church, then with the divisive rift between catholic and protestant thought; for now, we can disregard the influence of philosophy, so powerful in later times. This was powerfully joined by education system, operating under strong religious and social influence. Mathematics is neither a book enclosed in hard cover and clasped by brass buckles, with one only needing patience to skim through its pages; nor is it and underground shaft with treasures that can take long to obtain and are only filling a limited number of reefs and pockets; it is neither a soil with fertility depletable by repeated harvests; nor is it a continent or ocean that can have its surface mapped and outlines drawn: it is just as limitless as the cosmic space, seeing the latter as too narrow for its ambition; its possibilities are as limitless as the worlds that will forever present and multiply themselves before the eyes of astronomers. In early 13th century Dominican monk Jordanus

Nemorarius (Bolzano, 1851; Cantor, 1880; Riečan, 1988; Boyer & Merzbach, 2011) born in 1197 and living in the first half of the 13th century, produced an extensive mathematical work, influential in all respects; we focus here on several introductory phrases in his treatise on triangles (*De triangulis*) where he shows, quite strikingly, how far away he got from his Arabic and Greek models. Definitions it contains give impression of coming from 19th century from the likes of Dedekind or Bolzano (Bolzano, 1851; Cantor, 1880; Dedekind, 1888; Edwards, 1979; Perkins, 2012; Riečan, 1988). Here are some examples: *Continuity is indistinguishableness of boundary places, connected to a possibility of limitation. A point is the establishment of a simple continuity*. *An angle is created by intersection of two continuous shapes in the terminal point of their continuity*. …

All objections aside, such definitions from early 13th century are striking, as they show how the infinitesimal idea was being prepared already by the scholastics with all its patterns and difficulties. Our astonishment is no smaller when we read the works of Franciscan monk Thomas Bradwardine (ca. 1290 – 1349) serving some decades later in Oxford and appearing in works of several great doctors as the *Doctor Profundus*. Bradwardinus, who died as the Archbishop of Canterbury in the plague epidemic of 1349, wrote *inter alia* a treatise on continuity (*Tractatus de continuo*) containing many sentences that could be seen as excerpts from the doctrine of sets. He distinguishes permanent continuity (*continuum permanens*) in ex. lines, areas, or objects, and successive continuity (*continuum succesivum*) implemented by time or movement. The following sentences can also be found there: *Indivisible est, quodnunquam dividi potest. Punctus est indivisible situatum* (*The indivisible of time is, then, a moment. … Movement is a successive continuity measured in time*). Bradwardinus also distinguishes the issue of beginning and end. Thus, he naturally and necessarily comes to reflections on infinity (Coufal & Tobíšek, 2017; Boyer & Merzbach, 2011). He explains that a continuity can be assembled neither from a finite number of indivisible quantities, nor from infinite number of the indivisible. It only contains in itself infinitely much of the indivisible. Nicole Oresme (Coufal & Tobíšek, 2017) lived from ca. 1323 to 1382 and was a student, then a teacher, and finally the grand master of Collège de Navarre in Paris. He died as the bishop of Lisieux. One of his works, the *Tractatus de latitudinibus formarum*, is ultimately of interest. In his scholarship Oresme reach the "issue of tangents", the "differential quotient", and also the understanding that each point of a curve is characterized by the slope of the tangent to the curve passing this point.

3 Early Modernity

Let us continue on the Apennine peninsula, in Florence. It is not widely known, but one of the first successors of Archimedes in terms of creating the infinitesimal calculus was Galileo Galilei (1564 – 1642). And it is even less known who was his father. The famous father of an even more famous son, Vincenzo Galilei (1533 – 1591), lived in Florence and was one of the founders of opera (Riečan, 1988). The year is 1607 and the ducal court of Mantua is witness to the premiere of L'Orfeo by Monteverdi, a work still vivid in our times. Monteverdi, aged 40, is at the height of his creative prowess. The premiere is also attended by another artist of no smaller fame – the Flemish painter P. P. Rubens, then a 30-year old lad. The Duke pays little attention to the divine music, preferring the ladies present and sweets served. The Duke supports arts and the artists are thankful for that. In Italy of that times, an artist is barely more than a servant. And the nobility shows it, unequivocally. The fate of scholars differs little from that of artists. In the same period a famous son of a famous father, G. Galilei, request his salary at the Padova university to be raised. The Republic of Venice declines, arguing that mathematics is neither as needed as philosophy, nor as important as theology, and only provides pleasure to those who master it. Mathematics, therefore, is an art that does not pay well. Curiously, this idea still largely prevails today. But that view is false, as was shown by Galilei himself when he constructed a drawing compass that could be used to measure the weight of cannon balls, calculate compound interest, increase or decrease the scale of maps. Later he constructs a telescope and uses astronomic observations to produce new stellar maps necessary for sea travels. He is best known for his law on free-fall, and it is in this context that he was probably the first in Modernity to touch upon the issue of quadratures, and therefore of integration. Before Galilei and since 1613, the astrologist and mathematician at the court of Rudolf II in Prague, Johannes Kepler (1571 – 1630), used infinitely small quantities for quadratures, Kepler had little joy in childhood, but he struggled his way through to education, and following the studies of theology in Tübingen he became a teacher of mathematics in Graz, Austria. In 1596 he publishes his work *Mysterium Cosmographicum* where he tries to construct a heliocentric world-system. He sends this work to Galilei and to Tycho Brahe (1546 – 1601). Both scholars give him encouraging responses. Brahe, although himself an opponent of Copernicus' system, invites him to Prague. Kepler lives in Prague for 11 years (1600 – 1611), including one year of cooperation with Brahe whose place he takes over following the former's death. Using the records from Brahe's long-lasting astronomical observations he discovers his rules of movement of planets around the Sun. After the resignation of Rudolf II from the throne Kepler, having lost his patron (a proponent paying too little, but in important position), leaves Prague and becomes a teacher of mathematics in

Linz. Here he designs logarithmic tables, also helping him with astronomic calculations, and finishes the Rudolphine Tables he has been working on for 20 years. In his life-time, Kepler encountered numerous hardships and difficulties. For 6 years he defended (successfully in the end) his mother from being burnt at a stake. In Prague both his son and wife died. In 1613 he gets married again. For the wedding, it was necessary to procure wine, and Kepler finds inspiration in the measurement of the volume of a wine barrel. He calculates the volumes of more than 90 rotating objects. In 1615 he publishes *Nova Stereometria Doliorum Vinariorum* (New Stereometry of Wine Barrels) in which he calculates the volumes of objects created by rotation of conic sections around the axis situated in their plane. To do so, he used infinitesimal methods, and this treatise marked an important step towards the emergence of modern integration methods (Riečan, 1988). By the end of the first half of 17th century a full arsenal for systematic construction of integral and differential calculus was prepared. Isaac Newton (1643 – 1727) and Gottfried Wilhelm Leibniz (1646 – 1716) participated in its preparation. Leibniz made his discovery in Paris, building on French traditions in mathematics. Newton never left the British Isles; thus, he built his infinitesimal calculus mainly on the English tradition, represented by John Wallis (1616 – 1703) and Isaac Barrow (1630 – 1677). While Wallis was a self-taught mathematician, Barrow was a professional, having studied under Wallis. Wallis studied (in a distance learning, so to say) under Archimedes, Galilei, Descartes. Barrow studied ancient tongues in Cambridge, then mathematics in Oxford (under Wallis). He travelled a lot and, allegedly, such travels also got him among pirates. In 1659 he became a professor of Greek language in Oxford, later of geometry in London, and since 1663 of mathematics in Cambridge (it is here he taught Newton). He also studied Galilei and Descartes, but also Archimedes, whose works he prepared for publication and supplemented by commentary. When young Newton presented his discoveries of genius, Barrow recognized the singular talent of his student and yielded his professorship to him. That was in 1665. Eight years later Newton has all possible scientific degrees – not only he is a professor at Cambridge university, he is also a member of the Royal Society (the English academy of sciences, created in 1662). When Leibniz arrived in Paris in the spring of 1672, he was barely 26, but he had already graduated from two universities and become a university professor. Much like today, Paris was an attractive city and one of the hubs of the world of culture. Leibnitz was a prodigious child: aged 10 he read Latin and Greek classics, aged 13 he wrote 300 hexameters a day. Then he prepared for a lawyering career and, at 18, he tried to create combinatorics as a mathematical theory. Aged 20 he abandons a position at university in Nurnberg and moves on to Mainz, to serve the

Brandenburg Elector. He wanted to publish a perfect legal code, but got distracted by diplomatic missions and went to Paris where he met numerous mathematicians and physicists. Doubtless, this was also due to Christian Huygens (1629 – 1695), a composer and president of the French academy of sciences at the time. He took up the young man and, being an excellent physicist, astronomer, and mathematician, he inspired great interest in mathematical research in Leibniz. Despite the existential difficulties he had encountered since 1673 in Paris, in 1675 he discovered, independently from Newton, the differential and integral calculus. Unlike Newton (who used o) he used the symbol $\mathrm{d}x$ (differential of x) to mark an infinitely small quantity. His marking was so appropriate, it turned out, that it is still being used today. He handled the differentials as numbers, ex. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$, and the results are true even though it took generations of mathematicians to provide the proofs. In late 1676 he meets Newton in London, Baruch Spinoza in Amsterdam, and after that he spends the rest of his life as a librarian serving the Dukes of Hannover. His task was to work on the history of the Braunschweig-Lüneburgs. In connection to this work Leibniz travelled to Vienna, Rome and Naples in 1687 – 1690. And even before that, in 1684, he published (in Leipzig) the basic ideas of a new differential and integral calculus. He replied in spring to an article of his friend Tschiernhaus who published a work containing, *inter alia*, results of older debates with Leibniz, probably without Tschiernhaus realizing it. The main results were published by Leibniz in autumn of the same year. And even though this article marks the beginning of a new era, it was not written particularly well. Not even a mathematician as renowned as Jacob Bernoulli (1654 – 1705), one of the first and at the same time most important members of the Bernoulli scientific dynasty, could follow it properly. When Jacob could not reach the specific formulation of results, he addressed a letter to Leibniz, but received no answer. That was because Leibniz was travelling Italy at that time, and Jacob's letter probably waited for him in Hannover for three years. Jacob, however, was a thoughtful and brooding man. Having received no answer from Leibniz, in the three years that followed Jacob himself created the entire calculus. As the future showed, this had priceless value for the development of infinitesimal calculus. It is because Jacob introduced his brother Johann (1667 – 1748) into Leibniz's theory, thus creating the Leibniz scientific school. Allegedly, Leibniz said the following (Edwards, 1979, Perkins, 2012): "*I tried to write in such a way as to allow the student to always see the inner basics of the object studied, to allow him to identify the source of discovery, that is to say: so that he can delve into everything as if he himself had come up with it.*" This quote is slightly mean, if we realize how agonizing was Leibniz's text for Jacob

Bernoulli. What is certain is the following: after three years, Leibniz returns to Hannover. Upon arrival a pleasant surprise awaits him: the letter from Jacob Bernoulli. He is finally understood as a mathematician. In 1700 he contributed to establishment of the first German scientific society in Prussia. In 1706 a Scotsman John Keill charges Leibniz with plagiarism, stating that (Hairer & Wanner, 1996) "*Newton published his method in Acta Eruditorum under the pen name Leibniz, even though with changed symbols*." Leibniz objects, the decision is up to the English Royal Society, chaired by Newton. The decision of the Royal Society, given in 1712, states that "*Keill was in no way unjust towards Leibniz.*" At this time Leibniz suffers from the injustice, even though this English victory is truly Pyrrhic as further development of infinitesimal calculus shifts almost entirely to the continent. In 1714 the Duke Georg Ludwig of Hannover becomes the new king of England, not only refusing to allow Leibniz to go to London to defend himself before the Royal Society, but also not finding time for his visit in Hannover. Abandoned, Leibniz died in Hannover on 14 November 1716. One author even claims that it took one month to bury his body as he was seen as insufficiently orthodox by the protestant priests. And the Royal Society continued to judge him even after his death. Nevertheless, he was a man that enriched the world by numerous scientific discoveries. In times of cruel religious clashes and persecution Leibniz struggles for tolerance and atonement. In times of harsh wars and mass murders he works on implementing wide-ranging peace plans. In times of absolutist rule he presents an idea of society based on equality.

Conclusion

The development of mathematics is of course related to the development of the entire culture of which it is an inseparable component. Mathematics has always served humanity by providing it with effective ways of easing life and by creating the necessary pre-conditions for scientific and technical progress. It also served humanity by cultivating its thoughts, providing it with universal methods leading to understanding of nature, especially of the Man. Both these inseparable and intertwined noble tendencies reached their unrepeatable summit in the times of Baroque when differential and integral calculus was born. Just like it is impossible to surmount Johann Sebastian Bach and Georg Friedrich Händel in music, it is equally impossible to surmount Newton or Leibniz. The laying of foundations for calculus lead to significant enlargement of mathematical analysis. Infinitesimal calculus has entered all areas of scholarship and is today invaluable in biology, physics, chemistry, economics, sociology, technics, and in all scholarly disciplines where one quantity, such as speed or temperature, undergoes incessant changes. The methods of calculus have brought about a revolution, changing the view of the world.

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References

Aristotle. (1910-1952). *Works of Aristotle translated into English*. Ross, W. D. (ed.), Oxford Translation (12 vols.), Oxford.

Boyer, C. B., Merzbach, U. C. (2011). *A History of Mathematics*. New York : John Wiley & Sons. ISBN 978-04-70630-56-3.

Bolzano, B. (1851). *Paradoxien des Unendlichen*. Leipzig : Příhonský.

Cantor, M. B. (1880). *Vorlesungen über Geschichte der Mathematik* 1. Leipzig : B. G. Teubner.

Coufal, J., & Tobíšek, J. (2017). En Route for Infinity. In: *The 11th International Days of Statistics and Economics* (*MSED 2017*) [online]. Praha, 14.09.2017 – 16.09.2017. Slaný : Libuše Macáková, Melandrium, pp. 235 – 244. ISBN 978-80-87990-12-4 [cit. 2018-06-10]. Dostupné z WWW: https://msed.vse.cz/msed_2017/article/37-Coufal-Jan-paper.pdf.

Dedekind, R. (1888). *Was sind und was sollen die Zahlen*? Braunschweig : Vieweg.

Edwards, C. H. Jr. (1979). *The Historical Development of the Calculus*. New York, Heidelberg, Berlin: Springer Verlag.

Hairer, E., & Wanner, G. (1996). *Analysis by Its History*. New York : Springer.

Riečan, B. (1988). *Príbehy o integráloch*. Bratislava : SPN.

Perkins, D. (2012). *Calculus and Its Origins*. Washington : MAA Press.

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