LQ-MOMENTS AND THEIR USE IN ECONOMICS

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Abstract

LQ-moments represent a certain analogy of classical L-moments, which have been already widely applied in such areas of applied research as construction, meteorology and hydrology. Classical L-moments have number of various analogies, of which LQ-moments are one. This paper focuses on the use of LQ-moments in economics, in the given case in the construction of wage distribution models. The aim is to present the use and advantages of this alternative method of estimating the parameters of continuous probability distribution. The objective consists in presentation of the advantages of this method over the other methods of estimation of parameters (method of L-moments and maximum likelihood method), while the basic theoretical probability distribution is represented by three-parametric lognormal curves, whose parameters are estimated. The calculation of the sample LQ-moments and the construction of the various statistical characteristics of the level, variability, skewness and kurtosis of continuous probability distribution is also an important target of this research.

Key words: LQ-moments of probability distribution, sample LQ-moments, application of LQ-moments to lognormal distribution

JEL Code: C13, C18, C46

Introduction

The subject matter of the point estimation of parameters is still current in the specialized statistical literature. LQ-moments represent a more robust alternative to today's well-known L-moments and TL-moments. For example, (Mudholkar & Hutson, 1998) present a class of analogs of L-moments, labeled LQ-moments, obtained by replacing the expectations by functionals inducing quick estimators such as the median, Gastwirth estimator and the trimean. (Shabri & Jemain, 2006) develop improved LQ-moments that do not impose restrictions on the value of p and α such as the median, trimean or the Gastwirth but they explore an extended class of LQ-moments with consideration combinations of p and α values in the range 0 and 0.5. They propose weighted Kernel quantile estimator to estimate the quantile function. (Shabri & Jemain, 2010) also consider the four-parameter kappa distribution as a combination of the established distribution including generalized extreme value, generalized logistic, generalized Pareto and the Gumbel distribution. They develop the method of LQ-moments for the kappa distribution. (Shabri & Jemain, 2006) develop improved LQ-moments that do not impose restrictions on the value of p and α such as the median, trimean or the Gastwirth but they explore an extended class of LQ-moments with consideration combinations of p and α values in the range 0 and 0.5. They conduct Monte Carlo simulations to illustrate the performance of the proposed estimators of the three-parameter lognormal distribution. (Šimková & Picek, 2017) derive L-moments, LQ-moments and TL-moments of the generalized Pareto and generalized extreme-value distributions up to the fourth order and use the first three L-, LQ- and TL-moments to obtain estimators of their parameters. Performing a simulation study, they compare high-quantile estimates based on L-, LQ-, and TL-moments to the maximum likelihood estimate with respect to their sample mean squared error. (Ashour, El-Sheik & Abu El-Magd, 2015) derive the TL-moments of the exponentiated Pareto distribution and use the TL-moments to estimate the unknown parameters of this distribution. They also derive the LQ-moments with the three cases (trimean, median and Gastwirth) and use to estimate the unknown parameters of the exponentiated Pareto distribution. (Mudholkar & Natarajan, 2002) deal with measures for distributional classification and model selection and with L-measures and LQ-measures of skewness and kurtosis. They pronounce that L-measures and LQ-measures behave similarly, except that LQ-moments always exist, whereas L-moments exist only for distributions with finite expectations. (David & Nagaraja, 2003) also deal with some quick estimators of parameters or measurements of probability distributions. (Abu El-Magd, 2010) obtains TL-moments and LQ-moments of the exponentiated generalized extreme value distribution and uses to estimate the unknown parameters of the exponentiated generalized extreme value distribution. He obtains many special cases such as the L-moments, LH-moments and LL-moments. (Deka, Borah & Kakaty, 2009) determine the best fitting distribution to describe the annual series of maximum daily rainfall data for the period 1966 to 2007 of nine distantly located stations in North-East India. Generalized extreme value distribution, generalized logistic distribution, generalized Pareto distribution, lognormal distribution and Pearson distribution are fitted for this purpose using the methods of L-moments and LQ-moments. (Zin, Jemain & Ibrahim, 2009) determine the best fitting distribution to describe the annual series of maximum daily rainfall from 1975 to 2004 for 50 rain gauge stations in Peninsular Malaysia based on L-moments and LQ-moments. They consider generalized extreme value, generalized Pareto, generalized logistic, lognormal and Pearson distributions, the estimation of parameters of these distributions is determined using the L-moments and LQ-moments. (Bhuyan & Borah, 2011) use five probability distributions for the LQ-moments: generalized extreme value, generalized logistic, generalized Pareto, lognormal and Pearson type III. (Zaher, El-Sheik & Abu El-Magd, 2014) obtain the fuzzy least-squares estimator for the two-parameter Pareto distribution and compare the fuzzy estimator with different types of estimators. They obtain the trimmed linear moments, linear moments and linear quantile moments formulas for the two-parameter Pareto distribution and the TL-moments estimator, L-moments estimator and LQ-moments estimator for the Pareto distribution. (Zin & Jemain, 2008) start point for further research in quantile application such as in parameter estimation using LQ-moments method. Thirteen methods of non-parametric quantile estimation are here applied on six types of extreme distributions and their efficiencies. (Kandeel, 2015) presents an overview for recent works in L-moments TL-moments and LQ-moments.

1 LQ-moments

L-moments, certain linear functions of the expected values of order statistics have been already widely used in such areas of applied research as construction, meteorology and hydrology. A class of their analogues labeled LQ-moments is presented in this research, which are obtained by substituting expected values by functionals that induce the quick estimators such as median, Gastwirth and trimean. LQ-moments always exist, and they are easier to evaluate and estimate. Skewness and kurtosis measurements based on LQ-moments can be used as more appropriate and efficient alternatives to traditional beta coefficients. The simplicity of their analyzes is demonstrated in terms of asymptotic distribution of estimators of LQ-moments, LQ-skewness and LQ-kurtosis. In the statistical literature, the application of extreme values in flood data analysis is discussed in hydrology and several potential applications are here outlined.

1.1 LQ-moments of probability distribution

Let X_1, X_2, …, X_n is a random sample from a continuous distribution with the distribution function $F_X(\cdot)$, with the quantile function $Q_X(u) = F_X^{-1}(u)$ and let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ represents order statistics. Then the r-th L-moment λ_r is given

$$\lambda_r = r^{-1} \cdot \sum_{k=0}^{r-1} (-1)^k \cdot \binom{r-1}{k} \cdot E(X_{r-k:r}), \quad r = 1, 2, \dots . \tag{1}$$

We define the r-th LQ-moment ξ_r analogously

$$\xi_r = r^{-1} \cdot \sum_{k=0}^{r-1} (-1)^k \cdot \binom{r-1}{k} \cdot \tau_{p,\alpha}(X_{r-k:r}), \quad r = 1, 2, \dots , \tag{2}$$

where $0 \le \alpha \le ½$, $0 \le p \le ½$ and

$$\tau_{p,\alpha}(X_{r-k:r}) = pQ_{X_{r-k:r}}(\alpha) + (1-2p)Q_{X_{r-k:r}}(1/2) + pQ_{X_{r-k:r}}(1-\alpha) . \tag{3}$$

This is evident from equations (1) and (2) that expected value $E(\cdot)$ at point $\tau_{p,\alpha}(\cdot)$ in equation (2) defines L-moments. Another possible generalization of the L-moments including the replacement of the expected value in equation (1) is possible by TL-moments.

The linear combination $\tau_{p,\alpha,}$ defined by equation (3) is a quick measure of level of the random distribution of the order statistic $X_{r-k:r}$. Candidates for $\tau_{p,\alpha}$ include functionals generating common quick estimators, i.e.

Median $\quad Q_{X_{r-k:r}}\left(\frac{1}{2}\right),$ (4)

Trimean $\quad Q_{X_{r-k:r}}\left(\frac{1}{4}\right)\Big/4 + Q_{X_{r-k:r}}\left(\frac{1}{2}\right)\Big/2 + Q_{X_{r-k:r}}\left(\frac{3}{4}\right)\Big/4,$ (5)

Gastwirth $\quad 0{,}3Q_{X_{r-k:r}}\left(\frac{1}{3}\right) + 0{,}4Q_{X_{r-k:r}}\left(\frac{1}{2}\right) + 0{,}3Q_{X_{r-k:r}}\left(\frac{2}{3}\right).$ (6)

When sampling from a normal distribution, the Gastwirth based LQ-estimator is the most efficient among the possibilities given by the equations (4)−(6). The first four LQ-moments of the random variable X are usually used in the most practical applications involving LQ-moments, for example probability density classification and parametric estimation.

$$\xi_1 = \tau_{p,\alpha}(X), \tag{7}$$

$$\xi_2 = \frac{1}{2}\left[\tau_{p,\alpha}(X_{2:2}) - \tau_{p,\alpha}(X_{1:2})\right], \tag{8}$$

$$\xi_3 = \frac{1}{3}\left[\tau_{p,\alpha}(X_{3:3}) - 2\tau_{p,\alpha}(X_{2:3}) + \tau_{p,\alpha}(X_{1:3})\right], \tag{9}$$

$$\xi_4 = \frac{1}{4}\left[\tau_{p,\alpha}(X_{4:4}) - 3\tau_{p,\alpha}(X_{3:4}) + 3\tau_{p,\alpha}(X_{2:4}) - \tau_{p,\alpha}(X_{1:4})\right]. \tag{10}$$

It is clear that location measures $\tau_{p,\alpha}(\cdot)$ exit for any random variable X. Therefore, the r-th LQ-moment always exists and it is unique, if the distribution function $F_X(\cdot)$ is continuous. In addition, the evaluation of the LQ-moments of any continuous distribution is simplified using the following statement.

If $Q_X(\cdot) = F_X^{-1}(\cdot)$ is the quantile function of random variable X, then the quick measure of level defined by equation (3) is equivalent to equation

$$\tau_{p,\alpha}(X_{r-k:r}) = p\,Q_X[B_{r-k:r}^{-1}(\alpha)] + (1-2p)\,Q_X[B_{r-k:r}^{-1}(1/2)] + p\,Q_X[B_{r-k:r}^{-1}(1-\alpha)], \tag{11}$$

where $B_{r-k:r}^{-1}(\alpha)$ means the corresponding α-th quantile of beta distributed random variable with parameters $r-k$ and $k+1$.

The coefficients of skewness $\sqrt{\beta_1}$ and kurtosis β_2 have played an important role in classifying statistical distributions, when constructing appropriate distribution models, and estimating parameters. Due to their usefulness and due to some shortcomings of $\sqrt{\beta_1}$ and β_2, for example, their occasional non-existence, alternative measurements of skewness and kurtosis have been defined. They include various classical measures and relatively recent measurements as

$$\gamma_U(F) = [F^{-1}(1-u) + F^{-1}(u) - 2m_F]/[F^{-1}(1-u) + F^{-1}(u)].$$

Quantile based measurements of kurtosis for symmetric distributions include

$$[Q(0{,}75+u)+Q(0{,}75-u)-2Q(0{,}75)]/[Q(0{,}75+u)-Q(0{,}75-u)],\ 0\le u<1/4$$

and

$$[Q(0{,}5+u)-Q(0{,}5-u)]/[Q(0{,}75)-Q(0{,}25)],\ 0\le u<1/2.$$

The ratios τ_3 and τ_4 based on L-moments and called L-skewness and L-kurtosis are defined

$$\tau_3=\frac{\lambda_3}{\lambda_2}\quad\text{and}\quad\tau_4=\frac{\lambda_4}{\lambda_2}, \tag{12}$$

and they offer a new alternative to $\sqrt{\beta_1}$ and β_2. It is proved that τ_3 meets the convex arrangement and τ_4 maintains van Zwet's symmetrical arrangement.

The skewness and kurtosis measurements η_3 and η_4 based on LQ-moments are called LQ-skewness and LQ-kurtosis, and they are defined

$$\eta_3=\frac{\xi_3}{\xi_2}\quad\text{and}\quad\eta_4=\frac{\xi_4}{\xi_2}. \tag{13}$$

It is necessary to mention that LQ-skewness and LQ-kurtosis are invariant in terms of location and scale and they exist for all distributions. However, the analogies of the other properties of τ_3 and τ_4 mentioned above remain unexplored for η_3 and η_4. The behavior of LQ-skewness η_3 and LQ-kurtosis η_4 is now discussed by their comparison with their established counterparts.

Another ratio measurement useful for comparing the distributions with the usual origin and scale is the analogy of coefficient of variation

$$\eta_2=\frac{\xi_2}{\xi_1}, \tag{14}$$

where equations (7) and (8) represent ξ_1 and ξ_2. It is common practice in the modeling of survival to draw $\sqrt{b_1}$ versus sample coefficient of variation

$$\hat{\gamma} = \frac{s}{\bar{x}} \tag{15}$$

in the plane $(\sqrt{\beta_1}, \gamma)$ to verify the model selection.

1.2 Sample LQ-moments

The LQ-moments can be estimated directly by estimating the quantiles of the order statistics in combination with the equation (11) in their definition. The simplest quantile estimator suitable for this purpose is a quantile estimator based on linear interpolation in the usual statistical software packages. However, we can use any alternative estimator of quantiles.

Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the sample order statistics, then the quantile estimator of $Q(u)$ is given by the relationship

$$\hat{Q}_X(u) = (1-\varepsilon)\, X_{[n'u]:n} + \varepsilon\, X_{[n'u]+1:n}, \tag{16}$$

where $\varepsilon = n'u - [n'u]$ a $n' = n + 1$.

For random samples of sample size n, the r-th sample LQ-moment is given by the relationship

$$\hat{\xi}_r = r^{-1} \cdot \sum_{k=0}^{r-1} (-1)^k \cdot \binom{r-1}{k} \cdot \hat{\tau}_{p,\alpha}(X_{r-k:r}), \quad r = 1, 2, \ldots, \tag{17}$$

where $\hat{\tau}_{p,\alpha}(X_{r-k:r})$ is quick estimator of the level for distribution of order statistic $X_{r-k:r}$ in random sample of sample size r.

Namely, the first four sample LQ-moments from equation (17) are given

$$\hat{\xi}_1 = \hat{\tau}_{p,\alpha}(X), \tag{18}$$

$$\hat{\xi}_2 = \frac{1}{2}\left[\hat{\tau}_{p,\alpha}(X_{2:2}) - \hat{\tau}_{p,\alpha}(X_{1:2})\right], \tag{19}$$

$$\hat{\xi}_3 = \frac{1}{3}\left[\hat{\tau}_{p,\alpha}(X_{3:3}) - 2\,\hat{\tau}_{p,\alpha}(X_{2:3}) + \hat{\tau}_{p,\alpha}(X_{1:3})\right], \tag{20}$$

$$\hat{\xi}_4 = \frac{1}{4}\left[\hat{\tau}_{p,\alpha}(X_{4:4}) - 3\hat{\tau}_{p,\alpha}(X_{3:4}) + 3\hat{\tau}_{p,\alpha}(X_{2:4}) - \hat{\tau}_{p,\alpha}(X_{1:4})\right], \tag{21}$$

where quick estimator $\hat{\tau}_{p,\alpha}(X_{r-k:r})$ of the level of order statistic $X_{r-k:r}$ is given by relationship

$$\begin{aligned} \hat{\tau}_{p,\alpha}(X_{r-k:r}) &= p\,\hat{Q}_{X_{r-k:r}}(\alpha) + (1-2\,p)\,\hat{Q}_{X_{r-k:r}}(1/2) + p\,\hat{Q}_{X_{r-k:r}}(1-\alpha), \\ &= p\,\hat{Q}_X[B^{-1}_{r-k:r}(\alpha)] + (1-2p)\,\hat{Q}_X[B^{-1}_{r-k:r}(1/2)] + p\,\hat{Q}_X[B^{-1}_{r-k:r}(1-\alpha)], \end{aligned} \tag{22}$$

where $0 \le \alpha \le ½$, $0 \le p \le ½$, $B^{-1}_{r-k:r}(\alpha)$ is the α-th quantile of the random variable with beta distribution with parameters $r-k$ a $k+1$ and $\hat{Q}_X(\cdot)$ means an estimator using the linear interpolation given by the equation (16). Calculation of the sample LQ-moment $\hat{\xi}_r$ is simplified using the quantile $B^{-1}_{r-k:r}(\alpha)$ that can be easily obtained from statistical tables or statistical programs.

Explicit schemes for the calculation of LQ-moments are presented, when three the best known quick estimators, namely median ($p = 0, \alpha = \cdot$), trimean ($p = 1/4, \alpha = 1/4$) and Gastwirth (p = 0,3, α = 1/3), are used for $\hat{\tau}_{p,\alpha}(X_{r-k:r})$ given by equation (22). The estimation of the first four sample LQ-moments from equation (17) simplifies using the pyramid schemes.

Sample LQ-skewness and LQ-kurtosis are

$$\hat{\eta}_3 = \frac{\hat{\xi}_3}{\hat{\xi}_2} \quad \text{a} \quad \hat{\eta}_4 = \frac{\hat{\xi}_4}{\hat{\xi}_2}, \tag{23}$$

which can be used to identify η_3 and η_4 and to estimate parameters.

1.3 Large sample theory

Sample LQ-moments depend on the choice of the used quick estimator and on the quantile estimator used for the estimation. However, their asymptotic normality results from the theory of a large sample of linear functions of order statistics. In order to develop expressions for a large sample for the average and variance of the sample LQ-moments, we limit our attention to the class Q of the quantile functions Q meeting the following conditions.

1. The inverse function $Q_X(u) = F_X^{-1}(u)$ is exclusively defined for $0 < u < 1$.
2. $Q(\cdot)$ is twice differentiable on the interval (0, 1) with a continuous the second derivation $Q^{//}(\cdot)$ on the interval (0, 1).
3. $Q^{/}(\cdot) > 1$ for $0 < u < 1$.

Let $0 < u_1 < u_2 < \ldots < u_k < 1$ and we assume that conditions (1)–(3) above are fulfilled. Then $[\hat{Q}(u_1), \hat{Q}(u_2), ..., \hat{Q}(u_k)]$ is asymptotically normal with a vector of expected values $[Q(u_1), Q(u_2), \ldots, Q(u_k)]$ and with covariances

$$\sigma_{ij} = Cov[\hat{Q}(U_i), \hat{Q}(U_j)] = u_i\,(1-u_j)Q^{/}(u_i)Q^{/}(u_j)/n, \quad i \le j, \sigma_{ij} = \sigma_{ji}. \tag{24}$$

In order to create asymptotic expressions for covariances of LQ-moments, we will first derive

$$Cov[\hat{\tau}_{p,\alpha}(X_{r-k:r}), \hat{\tau}_{p,\alpha}(X_{s-l:s})],$$

which is a function dependent on six specific percentiles $u_1, u_2, \ldots, u_6$ used in calculating equation (24) overwritten into a set of six percentiles

$$B_{r-k:r}^{-1}(\alpha),\ B_{s-l:s}^{-1}(\alpha),\ B_{r-k:r}^{-1}(1/2),\ B_{s-l:s}^{-1}(1/2),\ B_{r-k:r}^{-1}(1-\alpha),\ B_{s-l:s}^{-1}(1-\alpha),$$

so that $0 < u_1 < u_2 < \ldots < u_6 < 1$, where $B_{r-k:r}^{-1}(\alpha)$ means the α-th quantile of random variable with beta distribution with parameters $r - k$ and $k + 1$. Then we can get $Cov[\hat{Q}(U_i), \hat{Q}(U_j)]$.

Covariance among the estimated quick estimators of order statistics is

$$Cov[\hat{\tau}_{p,\alpha}(X_{r-k:r}), \hat{\tau}_{p,\alpha}(X_{s-l:s})] =$$

$$= p\{pCov[\hat{Q}(u_1), \hat{Q}(u_2)] + (1-2p)Cov[\hat{Q}(u_2), \hat{Q}(u_3)] + pCov[\hat{Q}(u_2), \hat{Q}(u_5)] +$$
$$+ pCov[\hat{Q}(u_1), \hat{Q}(u_6)] + (1-2p)Cov[\hat{Q}(u_3), \hat{Q}(u_6)] + pCov[\hat{Q}(u_5), \hat{Q}(u_6)]\} + \quad (25)$$
$$+ (1-2p)\{pCov[\hat{Q}(u_1), \hat{Q}(u_4)] + (1-2p)Cov[\hat{Q}(u_3), \hat{Q}(u_4)] +$$
$$+ pCov[\hat{Q}(u_4), \hat{Q}(u_5)]\}.$$

Sample the r-th LQ-moment $\hat{\xi}_r$, $r = 1, 2, \ldots,$ has asymptotically normal distribution with expected value ξ_r and for $r \le s$, the covariances of LQ moments are given by equation

$$Cov(\hat{\xi}_r, \hat{\xi}_s) = \frac{1}{r\,s}\sum_{k=0}^{r-1}\sum_{l=0}^{s-1}(-1)^{k+l}\binom{r-1}{k}\binom{s-1}{l}\cdot Cov[\hat{\tau}_{p,\alpha}(X_{r-k:r}), \hat{\tau}_{p,\alpha}(X_{s-l:s})], \quad (26)$$

where $Cov[\hat{\tau}_{p,\alpha}(X_{r-k:r}), \hat{\tau}_{p,\alpha}(X_{s-l:s})]$ is given by an equation (25) and $u_1, u_2, \ldots, u_6$ are specified above. For $r = s$, we obtain the variance of the r-th sample LQ-moment $\hat{\xi}_r$.

As $n \to \infty$, the sample measures of LQ-skewness $\hat{\eta}_3$ and LQ-kurtosis $\hat{\eta}_4$ have two-dimensional normal distribution with a vector of the expected values (η_3, η_4) and

$$Var(\hat{\eta}_3) = Var(\hat{\xi}_3)/\hat{\xi}_2^2, \quad (27)$$

$$Cov(\hat{\eta}_3, \hat{\eta}_4) = [Cov(\hat{\xi}_3, \hat{\xi}_4) - \hat{\xi}_3 Cov(\hat{\xi}_2\ \hat{\xi}_4) - \hat{\xi}_4 Cov(\hat{\xi}_2, \hat{\xi}_3) + \hat{\xi}_3\hat{\xi}_4 Var(\hat{\xi}_2)]/\hat{\xi}_2^2, \quad (28)$$

$$Var(\hat{\eta}_4) = Var(\hat{\xi}_4)/\hat{\xi}_2^2, \quad (29)$$

where

$$Var(\hat{\xi}_r) = Cov(\hat{\xi}_r, \hat{\xi}_r)$$

and variance and covariances indicate the right side of the equation (26).

1.4 Application to normal distribution

We consider a random sample from the normal distribution and compare the use of the median, trimean and Gastwirth estimators to estimate LQ-skewness and LQ-kurtosis. Then the estimators $\hat{\eta}_3$ and $\hat{\eta}_4$ given by equation (23) have a common normal distribution with the corresponding expected value vectors

$$(0;\ 0{,}116),\ (0;\ 0{,}118)\ \text{a}\ (0;\ 0{,}117) \tag{30}$$

and with the covariance matrices

$$\boldsymbol{\Sigma}_{\text{MED}} = \frac{1}{n}\begin{pmatrix} 1{,}535 & 0 \\ 0 & 2{,}070 \end{pmatrix}, \boldsymbol{\Sigma}_{\text{TRI}} = \frac{1}{n}\begin{pmatrix} 0{,}824 & 0 \\ 0 & 0{,}381 \end{pmatrix} \text{a}\ \boldsymbol{\Sigma}_{\text{GAS}} = \frac{1}{n}\begin{pmatrix} 0{,}549 & 0 \\ 0 & 0{,}235 \end{pmatrix}. \tag{31}$$

We can see from equations (31) that $\hat{\eta}_3$ and $\hat{\eta}_4$ are asymptotically uncorrelated for each from the above-mentioned quick estimators. In addition, we can see from covariance matrices that we prefer the Gastwirth estimator against median and trimean estimators in terms of estimators of skewness and kurtosis in the case of large samples from almost normal distribution.

1.5 Application to lognormal distribution

LQ-estimators for three-parametric lognormal distribution behave similarly to L-moment estimators. We get the following expressions for LQ-moments of three-parametric lognormal distribution from the equations (7)–(9) and (13)

$$\xi_1 = \theta + \exp(\mu)\,\tau_{p,\alpha}(X_{1:1}), \tag{32}$$

$$\xi_2 = \frac{1}{2}\exp(\mu)\,[\tau_{p,\alpha}(X_{2:2}) - \tau_{p,\alpha}(X_{1:2})], \tag{33}$$

LQ-skewness coefficient can be calculated using

$$\eta_3 = \frac{\frac{1}{3}\left[\tau_{p,\alpha}(X_{3:3}) - 2\,\tau_{p,\alpha}(X_{2:3}) + \tau_{p,\alpha}(X_{1:3})\right]}{\frac{1}{2}\left[\tau_{p,\alpha}(X_{2:2}) - \tau_{p,\alpha}(X_{1:2})\right]}. \tag{34}$$

The LQ-parameter estimators $\hat{\mu}$, $\hat{\sigma}$ and $\hat{\theta}$ represent the solution of equations (7)–(9) in combination with equations (32)–(34) for μ, σ and θ, where we replace ξ_r with $\hat{\xi}_r$. Using the regression analysis, we obtain the following approximate relationship, which we can use for estimation of $\hat{\sigma}$ for $|\eta_3| \leq 1{,}0$ a $|k| \leq 2{,}64$

$$\hat{\sigma} = 2{,}1684\hat{\eta}_3 + 0{,}3967\,\hat{\eta}_3^3 + 0{,}1744\hat{\eta}_3^5 - 0{,}1015\hat{\eta}_3^7. \tag{35}$$

Once we obtain the value $\hat{\sigma}$, we can also obtain estimates $\hat{\mu}$ and $\hat{\theta}$ using equations (33) and (32).

1.6 Appropriateness of the model

It is also necessary to assess the suitability of constructed model or choose a model from several alternatives, which is made by some criterion, which can be a sum of absolute deviations of the observed and theoretical frequencies for all intervals

$$S = \sum_{i=1}^{k} |n_i - n\,\pi_i| \tag{36}$$

or known criterion χ^2

$$\chi^2 = \sum_{i=1}^{k} \frac{(n_i - n\,\pi_i)^2}{n\,\pi_i}, \tag{37}$$

where n_i are the observed frequencies in individual intervals, π_i are the theoretical probabilities of membership of statistical unit into the i-th interval, n is the total sample size of corresponding statistical file, $n \cdot \pi_i$ are the theoretical frequencies in individual intervals, $i = 1, 2, ..., k$, and k is the number of intervals.

The question of the appropriateness of the given curve for model of the distribution of wage is not entirely conventional mathematical-statistical problem in which we test the null hypothesis

H_0: The sample comes from the supposed theoretical distribution

against the alternative hypothesis

H_1: non H_0,

because in goodness of fit tests in the case of wage distribution we meet frequently with the fact that we work with large sample sizes and therefore the tests would almost always lead to the rejection of the null hypothesis. This results not only from the fact that with such large sample sizes the power of the test is so high at the chosen significance level that the test uncovers all the slightest deviations of the actual wage distribution and a model, but it also results from the principle of construction of the test. But practically we are not interested in such small deviations, so only gross agreement of the model with reality is sufficient and we so called "borrow" the model (curve). Test criterion χ^2 can be used in that direction only tentatively. When evaluating the suitability of the model we proceed to a large extent subjective and we rely on experience and logical analysis.

2 Results

The data base of the research consist in employees of the Czech Republic in the period of 2009–2016. There are a total set of all employees of the Czech Republic together and further the partial sets broken down by various demographic and socio-economic factors. The researched variable is the gross monthly wage in CZK (nominal wage). Data come from the official website of the Czech Statistical Office. The data were in the form of interval frequency distribution, since the individual data is not currently available. There were 328 wage distributions in total.

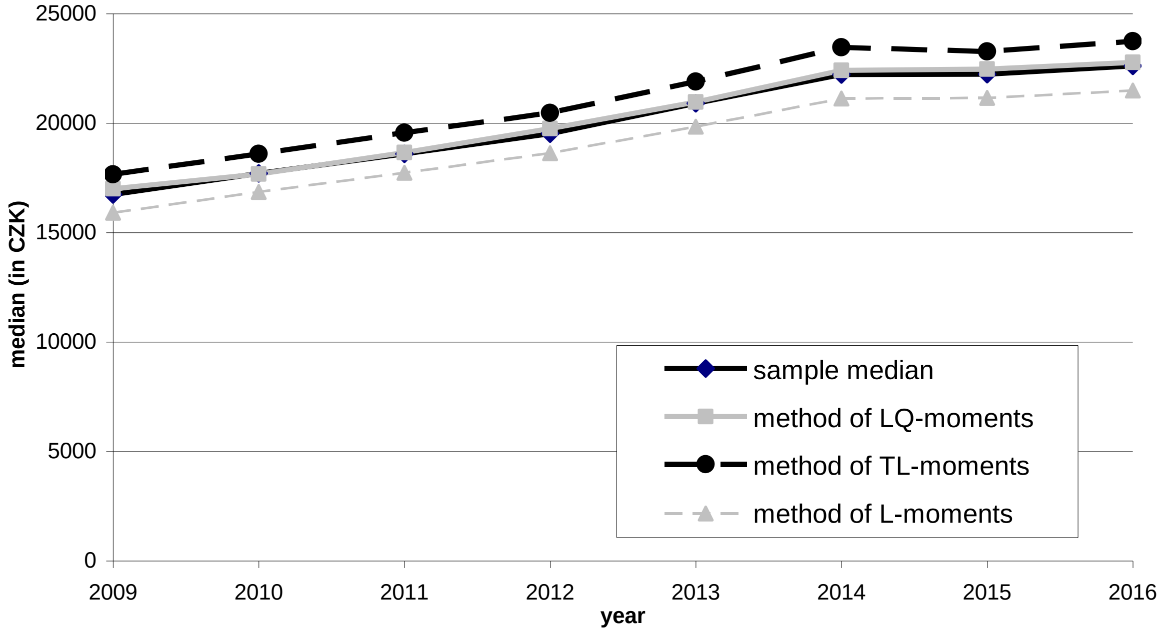
Table 1 presents parameter estimations obtained using the various three methods of point parameter estimation and the value of criterion *S* for the total wage distribution of the Czech Republic. This table describes approximately the research results of all 328 wage distributions. We obtained in total research that the method of LQ-moments provided the most accurate results in almost all cases of wage distribution with minor exceptions, deviations occur mainly at both ends of the wage distribution due to the extreme open intervals of interval frequency distribution. In the results of Table 1 for total sets of wage distribution of the Czech Republic in 2009–2016 method of LQ-moments always brings the most accurate results in terms of criterion *S*. In terms of research of all 328 wage distribution, method of TL-moments brought the second most accurate results in more than in half of the cases. Deviations occur again especially at both ends of the distribution. In the results of Table 1 method of TL-moments brought the second most accurate results in terms of all total sets of wage distribution of the Czech Republic in 2009–2016.

Tab. 1: Parameter estimations obtained using the various three methods of point parameter estimation and the value of criterion *S* for the total wage distribution of the Czech Republic

Year	Method								
	Method of LQ-moments			Method of TL-moments			Method of L-moments		
	Parameter estimation			Parameter estimation			Parameter estimation		
	μ	σ^2	θ	μ	σ^2	θ	μ	σ^2	θ
2009	9.059 747	0.630 754	9,065.52	9.017 534	0.608 369	7,664.46	9.741 305	0.197 395	2.07
2010	9.215 324	0.581 251	8,552.10	9.241 235	0.507 676	6,541.16	9.780 008	0.232 406	0.22
2011	9.277 248	0.573 002	8,872.54	9.283 399	0.515 290	6,977.45	9.833 604	0.228 654	0.27
2012	9.313 803	0.577 726	9,382.66	9.283 883	0.543 225	7,868.21	9.890 594	0.210 672	0.59
2013	9.382 135	0.680 571	10,027.84	9.387 739	0.601 135	7,902.64	9.950 263	0.268 224	0.16
2014	9.438 936	0.688 668	10,898.39	9.423 053	0.624 340	8,754.64	10.017 433	0.264 124	0.19
2015	9.444 217	0.703 536	10,640.53	9.431 478	0.631 013	8,684.51	10.019 787	0.269 047	0.20
2016	9.482 060	0.681 258	10,616.80	9.453 027	0.621 057	8,746.20	10.033 810	0.269 895	0.20
	Criterion *S*			Criterion *S*			Criterion *S*		
2009		108,437.01			133,320.79			248,331.74	
2010		146,509.34			248,438.78			281,541.41	
2011		137,422.05			231,978.79			311,008.23	
2012		149,144.68			216,373.24			325,055.67	
2013		198,670.74			366,202.87			370,373.62	
2014		206,698.93			357,668.48			391,346.02	
2015		193,559.55			335,999.20			359,736.37	
2016		202,367.04			323,851,84			389,542.21	

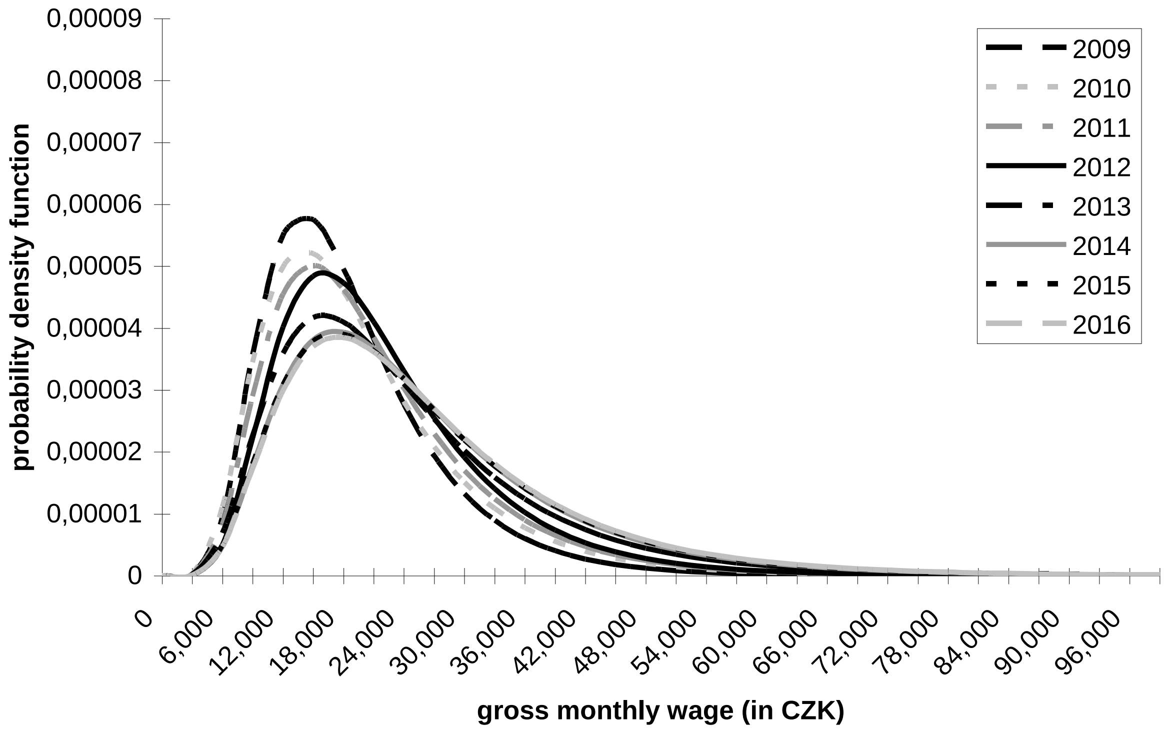
Source: Own research

Fig. 1 Development of sample and theoretical median of three-parametric lognormal curves with parameters estimated using three various methods of parameter estimation



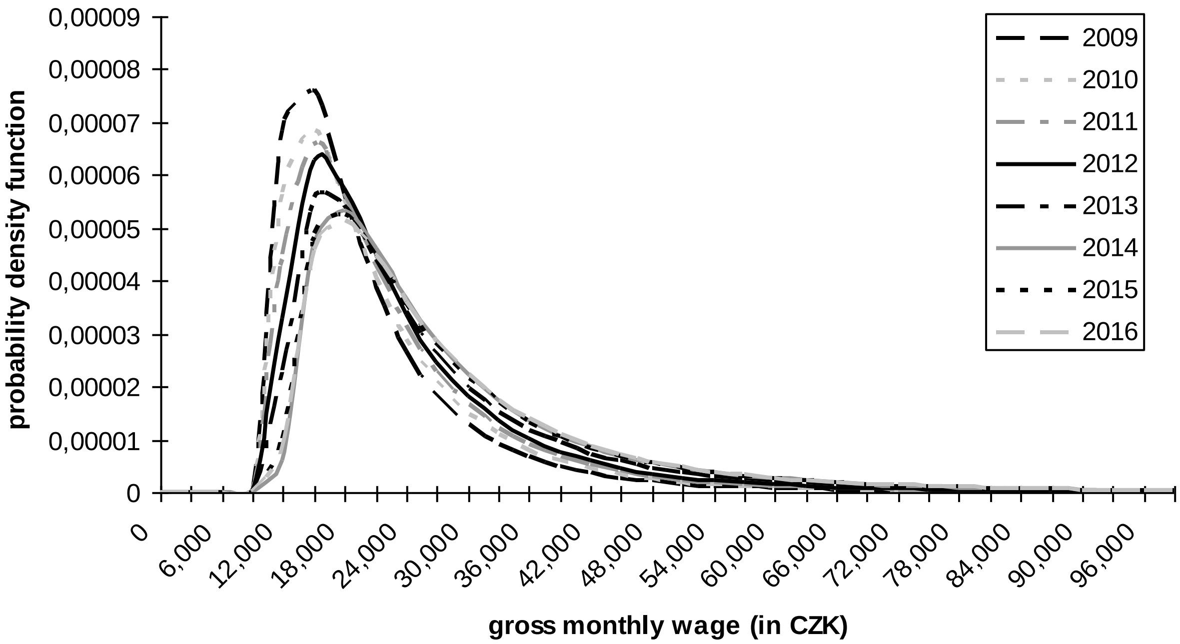
Source: Own research

Fig. 2: Development of probability density function of three-parametric lognormal curves with parameters estimated using the method of LQ-moments



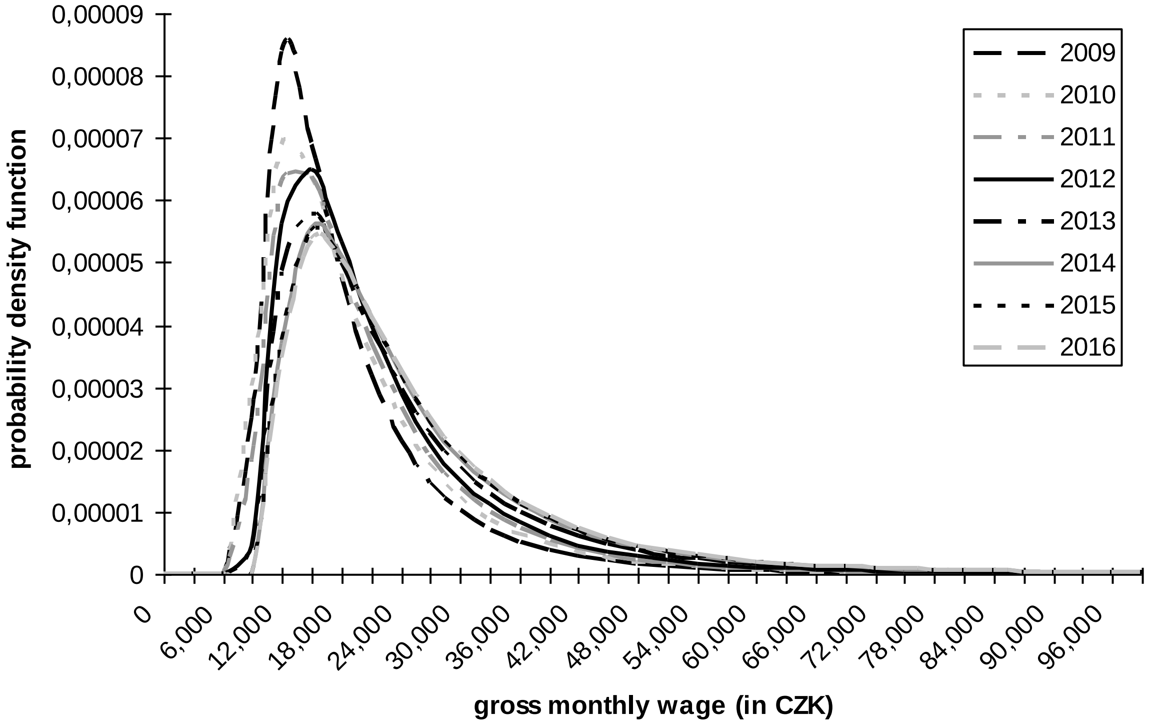
Source: Own research

Fig. 3: Development of probability density function of three-parametric lognormal curves with parameters estimated using the method of TL-moments



Source: Own research

Fig. 4: Development of probability density function of three-parametric lognormal curves with parameters estimated using the method of L-moments



Source: Own research

Overall, method of L-moments was the third in most cases in terms of accuracy of the results obtained (in all cases in Table 1). Figure 1 also gives some idea of the accuracy of the researched methods of point parameter estimation. This figure shows the development of the sample median of gross monthly wage for the total set of all employees of the Czech Republic together in the period 2009–2016 and the development of corresponding theoretical median of model three-parameter lognormal curves with parameters estimated by three various methods of point parameter estimation. We can observe from this figure that the curve characterizing the course of theoretical median of three-parameter lognormal distribution with parameters estimated using the method of LQ-moments adheres the most to the curve showing the development of the sample median. The other two curves articulating the development of the theoretical median of three-parameter lognormal curves with parameters estimated by method of TL-moments and by method of L-moments are relatively remote from the course of sample median of wage distribution.

Figures 2–4 represent the development of probability density function of three-parametrer lognormal curves with parameters estimated using the method of LQ-moments, method of TL-moments and method of L-moments. This is again a development of model distributions of the total wage distribution of the Czech Republic for all employees of the Czech Republic together in the period 2009—2016. We can see that the shapes of the lognormal curves with parameters estimated using the method of TL-moments and method of L-moments (Figures 3 and 4) are similar mutually and they are very different from the shape of three-parameter lognormal curves with parameters estimated by the method of LQ-moments (Figure 2).

Conclusion

Alternative category of moment characteristics of probability distributions was introduced here. There are the characteristics in the form of L-moments, TL-moments and LQ-moments. Accuracy of the methods of TL-moments and L-moments was compared with the accuracy of the method of LQ-moments using such criterion as the sum of all absolute deviations of the observed and theoretical frequencies for all intervals. Higher accuracy of the method of LQ-moments due to the method of TL-moments and to the method of L-moments was proved by studying of the set of 328 wage distributions. However, the advantages of the method of TL-moments to the method of L-moments were demonstrated here, too. The values of χ^2 criterion were also calculated for each wage distribution, but this test led always to the rejection of the null hypothesis about the supposed shape of the distribution due to the large sample sizes, which are typical for wage distribution.

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