BRIDGE ESTIMATOR AS AN ALTERNATIVE TO DICKEY-PANTULA UNIT ROOT TEST

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Abstract

Economic series may have unit roots. There are different methods to test unit roots in the literature. Recent studies show the advantages of using Bridge estimator for choosing the lag length of unit root tests and computing test statistics. This method penalizes the parameters of the model with a positive shrinkage parameter. For a suitable choice of the shrinkage parameter, the limiting distributions of the Bridge estimator can have positive probability mass at 0 when the true value of the parameter is zero for stationary series. Therefore, the Bridge estimator provides a method that can be used for the model selection and estimation simultaneously. It is known in the literature that Bridge estimator does the model selection while simultaneously distinguishing between stationary and unit root models for series with a unit root as well. In this study, we propose Bridge estimator to determine the integration order of series with more than one unit root. In order to do this, we apply the Bridge estimator to Dickey-Pantula model. We evaluate the size and power of the Bridge estimator with a simulation study and compare our proposed estimator with the existent Dickey-Pantula test.

Key words: Unit Root Test, Oracle Property, Model Selection, Bridge Estimator, LASSO

JEL Code: C12, C15, C22.

Introduction

Stationarity is a very important assumption in many econometric techniques. However, in the practice, most of the economic time series are nonstationary and ignoring the nonstationarity causes unreliable results. The augmented Dickey-Fuller (ADF, Dickey and Fuller, 1979) tests are commonly applied to check whether the series is stationary or not. Some researchers apply ADF test after taking the difference of the series to identify the order of integration. However, ADF test assumes that the series has single unit root (Dickey and Fuller, 1981). According to Dickey and Pantula (1987), applying ADF test for differenced series causes some statistical problems. For this reason, it is suggested to use Dickey-Pantula unit root test to determine the order of integration of variables when there is more than one unit root. Recently, Caner and

Knight (2013) propose Bridge Estimator to do model selection while simultaneously determine whether the series is $I(1)$ or $I(0)$. They show that Bridge estimator outperforms ADF tests in terms of size and power. In this study we describe Bridge estimator as an alternative to Dickey-Pantula unit root test. We extend Caner and Knight's (2013) approach to this model and compare the size and power of these approaches in a simulation study.

1 Dickey-Pantula Unit Root Test

Dickey and Pantula (1987), propose a proper sequence of statistical method to investigate series having more than one unit root. Let the time series $\{Y_t\}$ satisfy

$$Y_t = \alpha_1 Y_{t-1} + \alpha_2 Y_{t-2} + \alpha_3 Y_{t-3} + e_t , \qquad (1)$$

where $\{e_t\}$ is a sequence of iid random variables with mean 0 and variance σ^2. Let m_1, m_2 and m_3 denote the roots of characteristic equation $m^3 - \alpha_1 m^2 - \alpha_2 m - \alpha_3 = 0$. Assume that $1 \geq |m_1| \geq |m_2| \geq |m_3|$. Consider the following four hypotheses:

(a) $H_0 : |m_1| < 1$, (Y_t is stationary - $I(0)$);

(b) $H_1 : m_1 = 1, |m_2| < 1$, (Y_t has one unit root - $I(1)$);

(c) $H_2 : m_1 = m_2 = 1, |m_3| < 1$, (Y_t has two unit roots - $I(2)$);

(d) $H_3 : m_1 = m_2 = m_3 = 1$, (Y_t has three unit roots - $I(3)$).

That is under the hypothesis H_d, the dth difference of Y_t is essentially stationary (Dickey and Pantula, 1987). Now consider a reparameterization of model (1):

$$X_t = \theta_1 Y_{t-1} + \theta_2 Z_{t-1} + \theta_3 W_{t-1} + e_t , \qquad (2)$$

where $Z_t = Y_t - Y_{t-1}$, $W_t = Z_t - Z_{t-1}$, $X_t = W_t - W_{t-1}$ are first, second, and third differences of the sequence Y_t, respectively. Dickey and Pantula (1987) note that $\alpha_1 = 3 + \theta_1 + \theta_2 + \theta_3$, $\alpha_2 = -(3 + \theta_2 + 2\theta_3)$, $\alpha_1 = 1 + \theta_3$ and that $\theta_1 = -(1-m_1)(1-m_2)(1-m_3)$, $\theta_2 = -2\theta_1 - (1-m_1)(1-m_2) - (1-m_2)(1-m_3) - (1-m_3)(1-m_1)$, $\theta_3 = m_1 m_2 m_3 - 1$ to write previous four hypotheses in terms of θs as follows:

(a) $H_3 : \theta_1 = \theta_2 = \theta_3 = 0$;

(b) $H_2 : \theta_1 = \theta_2,\ \theta_3 < 0$;

(c) $H_1 : \theta_1 = 0,\ \theta_2 < 0,\ \theta_3 < 0$;

(d) $H_0 : \theta_1 < 0,\ \theta_2 < 0,\ \theta_3 < 0$.

Given n observations, it is possible to regress X_t on Y_{t-1}, Z_{t-1} and W_{t-1} to get ordinary least squares (OLS) estimates of $\hat{\theta}_1$, $\hat{\theta}_2$, and $\hat{\theta}_3$ and the corresponding t statistics $t_{1,n}(3)$, $t_{2,n}(3)$, $t_{3,n}(3)$. However, Dickey and Pantula argue that it may not be appropriate to use standard t statistics. They show that when the model in (2) holds with $\{e_t\}$ a sequence of iid $(0,\sigma^2)$ random variables, the t statistic $t_{1,n}(3)$ for testing one unit root has different asymptotic distributions depending on the number of unit roots present. In addition to this, they show that a sequential procedure based on the t statistics $t_{i,n}(3)$ is not consistent.

In order to solve this problem, Dickey and Pantula (1987) suggest a sequential procedure based on pseudo t statistics $t^*_{i,n}(p)$, where $t^*_{i,n}(p)$ is the t statistic for the coefficient of $(1-B)^{i-1}Y_{t-1}$ in the regression of $(1-B)^p Y_t$ on $(1-B)^{i-1}Y_{t-1}$, $(1-B)^i Y_{t-1}, \ldots,$ $(1-B)^{p-1}Y_{t-1}$. Their sequential procedure based on Dickey-Fuller τ statistics is as follows:

1. Reject the hypothesis H_3 of three unit roots and go to step 2 if $t^*_{3,n}(3) \le \hat{\tau}_{n,\alpha}$ where $\hat{\tau}_{n,\alpha}$ was given by Fuller (1976).
2. Reject the hypothesis H_2 of exactly two unit roots and go to step 3 if in addition to $t^*_{3,n}(3) \le \hat{\tau}_{n,\alpha}$ you also find that $t^*_{2,n}(3) \le \hat{\tau}_{n,\alpha}$.
3. Reject the hypothesis H_1 of exactly one unit root in favor of the hypothesis H_0 of no unit roots if $t^*_{i,n}(3) \le \hat{\tau}_{n,\alpha}$ ($i = 1,2,3$).

They show that the sequential procedure based on $t^*_{i,n}(p)$ is a consistent level-α procedure. Moreover, the probability that the procedure chooses the true number of unit roots present converges to $1-\alpha$ when the series has at least one unit root and to 1 when the series is stationary. Dickey and Pantula (1987) extend their theoretical comparisons with a numerical example and simulation study. They show that their method is superior to Dickey-Fuller t statistics and Hasza-Fuller F statistics in terms of size and power.

2 BRIDGE Estimator to Determine Integration Order of Series

Let a linear regression model with n observations and p independent variables

$$y = X\beta + e \qquad (3)$$

where $y : n \times 1$ is the vector of dependent variable, $X : n \times p$ is the matrix of independent variables, $\beta : p \times 1$ is the vector of unknown parameters and $e : n \times 1$ is the vector of unobservable error terms with $e \sim \left(0, \sigma^2 I_n\right)$. OLS minimizes sum of squared residuals:

$$\hat{\beta} = \min\left\{(y - X\beta)'(y - X\beta)\right\} \text{ and } \hat{\beta} = (X'X)^{-1}X'y. \qquad (4)$$

Recently, Bridge / Lasso type estimators are proposed to penalize model parameters while minimizing the sum of squared residuals as follows:

$$\hat{\beta}^* = \min\left\{(y - X\beta)'(y - X\beta) + \lambda \sum_j \left|\beta_j\right|^\gamma\right\}, \qquad (5)$$

where $\lambda > 0$ is the tuning parameter and $\gamma > 0$ is the shrinkage parameter. Frank and Friedman (1993), and Fu (1998) call these estimators as Bridge estimators. Tibshirani (2011) defines the special case ($\gamma = 1$) as Least Absolute Shrinkage (LASSO). Fu (1998) use different shrinkage parameters and compare Bridge, OLS, LASSO and Ridge with a simulation study. Fu and Knight (2000) establish the limit law for LASSO. These estimators shrink the estimates of zero parameter(s) to zero with probability converging to 1 when $0 < \gamma \leq 1$. This is called the oracle property. Fan and Li (2001) show that these methods simultaneously select the model and estimate the coefficients. Huang, Horowitz and Ma (2008) extend Knight and Fu's (2000) study to investigate the asymptotic properties of Bridge estimators for stationary series. Caner (2009) consider LS and Generalized Method of Moments (GMM) based Lasso estimators, respectively and finds that in small samples Lasso estimator have smaller mean square error and bias than the ones chosen by Akaike Information Criterion (AIC), Schwartz Criterion (SC) and sequential testing procedures.

In a recent paper, Caner and Knight (2013) find the limiting distributions of Bridge estimators when the series is $I(1)$. Their proposed method is a penalized version of ADF test. This method does the model selection while simultaneously distinguishing between stationary and unit root models. By doing so, they eliminate the two-step procedure of model selection (i.e., deciding the deterministic components and choosing the lag length) and then applying a unit root test. Note that, Caner and Knight's (2013) method is not a unit root test since there is no testing step in their procedure. Instead of that, they propose Bridge estimator to choose between unit root and stationary models to decide whether the series is integrated or not. Their simulations show that this method select the optimal lag length and unit root simultaneously and makes a substantial difference in terms of size and power when it is compared with the existing unit root tests with lag selection methods. For the ADF model, Caner and Knight (2013) show that it is possible to estimate the coefficient of lagged dependent variable consistently regardless of stationarity or nonstationarity when $0 < \gamma < 1/2$. In their simulations they show that the method performs well when $\gamma = 1/4$.

In this paper, we extend Caner and Knight's (2013) method to Dickey-Pantula model. Since Bridge estimator performs well in model selection, we hope that our approach leads to some gain in terms of size and power over Dickey-Pantula test, especially when the series contains more than one unit root. In order to do this, we penalize the parameters of (2) while minimizing the sum of squared residuals as follows:

$$\hat{\theta}^* = \min\left\{\sum_t \left(X_t - \theta_1 Y_{t-1} - \theta_2 Z_{t-1} - \theta_3 W_{t-1}\right)^2 + \lambda \sum_j \left|\theta_j\right|^\gamma\right\} \qquad (6)$$

Note that, (6) is a nonlinear optimization problem since it involves the powers of model parameters. By following Caner and Knight's (2013) suggestion, we use $\gamma = 1/4$ in (6). In order to determine the tuning parameter λ, we use Caner and Knight's Modified Bayesian Information Criteria (MBIC),

$$MBIC(\lambda) = \left\{\log\left[\sum_t \left(X_t - \hat{\theta}_1 Y_{t-1} - \hat{\theta}_2 Z_{t-1} - \hat{\theta}_3 W_{t-1}\right)^2 \Big/ n\right] + \frac{\log n}{n}\left|S_\lambda\right|\right\} \qquad (7)$$

where $\hat{\theta}_1$, $\hat{\theta}_2$, and $\hat{\theta}_3$ are Bridge estimates that correspond to a specific choice of λ and $\left|S_\lambda\right|$ is the number of nonzero Bridge estimates. The estimate of λ is obtained from

$$\hat{\lambda} = \arg\min_\lambda MBIC(\lambda). \qquad (8)$$

For selection of the values of λ, we follow Caner ad Knight (2013) and use the set $\lambda = \{0.1, 1, 10, 100\}$.

However, Bridge estimators of the zero components may take both positive and negative values. In order to solve this problem, the components of the Bridge estimator that are close to zero are forced to be exactly zero. This is done by comparing the Bridge estimator with a hard thresholding parameter $c_1 > 0$. Caner and Knight (2013) suggest to set the coefficient of the lagged dependent variable to be zero when its Bridge estimate is greater than $-c_1/n$. Similarly, we set $\hat{\theta}_i = 0$ when $\hat{\theta}_i > -c_{1(i)}/n$ where $c_{1(i)}$ is the hard thresholding parameter for $\hat{\theta}_i$ ($i = 1,2,3$). Caner and Knight (2013) choose c_1 that corresponds to 5%

wrong model selection when the true data generating process (DGP) is nonstationary and denote this as $c_{1,0.05}$. This is similar to create the critical value of a test statistic by setting $\alpha = 0.05$. By following their approach, we apply the same method to choose $c_{1(i),0.05}$ values for $\hat{\theta}_i$ ($i = 1,2,3$) in our simulations.

Once Bridge estimates of (2) are estimated by (6), we can check the nonzero coefficients of $\hat{\theta}_i$ ($i = 1,2,3$). Similar to Dickey-Pantula test, we can apply following sequential decision criteria to determine integration order of series:

1. Reject the hypothesis H_3 of three unit roots and go to step 2 if $\hat{\theta}_3 < 0$.
2. Reject the hypothesis H_2 of exactly two unit roots and go to step 3 if in addition to $\hat{\theta}_3 < 0$ you also find that $\hat{\theta}_2 < 0$.
3. Reject the hypothesis H_1 of exactly one unit root in favor of the hypothesis H_0 of no unit roots if $\hat{\theta}_i < 0$ ($i = 1,2,3$).

Following these steps, we can determine the integration order of series. Please note that this is not a sequential estimation procedure as in Dickey-Pantula test. Our method involves estimation of the parameters simultaneously and then sequentially checking the nonzero coefficients starting from $\hat{\theta}_3$ to $\hat{\theta}_1$. That is why we call this method as a sequential decision criteria. One can also check the nonzero coefficients simultaneously but this might lead to inconclusive decisions (for instance $\theta_3 = \theta_1 = 0$ but $\theta_2 \neq 0$). Our simulations show that this is a very unlikely case. However to avoid inconclusive decisions that might happen, we suggest the sequential process given above.

3 Simulation Experiment

In this section, we compare Dickey and Pantula's (1987) sequential procedure based on $t^*_{i,n}(p)$ with the proposed Bridge estimator. However, as pointed out by Caner and Knight (2013), Bridge is a model selection/estimation setup and there is no testing procedure. In order to compare Bridge with Dickey-Pantula, we can use the correct model selection percentages. The idea is to compare estimated integration of orders when the true DGP contains unit root(s) and stationary. Since we fix the wrong model selection by using exact critical values and hard thresholding parameters, wrong and correct model selection percentages denote the size and size-adjusted powers. Therefore, we first obtain exact critical values for Dickey-Pantula test and hard thresholding parameters for Bridge estimator. Then we obtain size-adjusted powers to compare aforementioned methods. We use MATLAB to perform simulations. We use the following DGP to generate data:

$$Y_t = \alpha_1 Y_{t-1} + \alpha_2 Y_{t-2} + \alpha_3 Y_{t-3} + e_t, t = 1,2,\ldots,100. \quad (9)$$

where $n = 100$, e_t are independently and randomly generated from $N(0,1)$ and the initial values Y_0, Y_{-1}, and Y_{-2} are set to zero. The parameters α_1, α_2, and α_3, are computed from the triplets of roots listed in the left column of Table 1. This allows us to control the integration order of Y_t by varying m_1, m_2, and m_3.

To obtain exact critical values for Dickey-Pantula test statistics, we use roots $m_1 = m_2 = m_3 = 1$ ($d = 3$), $m_1 = m_2 = 1, m_3 = 0$ ($d = 2$), and $m_1 = 1, m_2 = m_3 = 0$ ($d = 1$) for $t^*_{3,n}(3)$, $t^*_{2,n}(3)$, and $t^*_{1,n}(3)$, respectively. Then for each case, we obtain 5% empirical critical values with 2 000 replications. These are -1.9107, -1.8818, and -1.8396 for $t^*_{3,n}(3)$, $t^*_{2,n}(3)$, and $t^*_{1,n}(3)$, respectively. These exact critical values are used instead of $\hat{\tau}_{n,\alpha}$ in the sequential Dickey-Pantula test. The same DGP is applied to our proposed Bridge estimator to obtain hard thresholding parameters $c_{1(i),0.05}$ ($i = 1,2,3$). We choose $c_{1(3),0.05}$, $c_{1(2),0.05}$, and $c_{1(1),0.05}$ so that wrong model selection percentages are set to 5% for $m_1 = m_2 = m_3 = 1$ ($d = 3$), $m_1 = m_2 = 1, m_3 = 0$ ($d = 2$), and $m_1 = 1, m_2 = m_3 = 0$ ($d = 1$), respectively. These hard thresholding parameters are $c_{1(3),0.05} = 13$, $c_{1(2),0.05} = 9.4$, and $c_{1(1),0.05} = 2.753$.

In order to obtain size and size-adjusted powers, we use the same DGP to generate data with the roots listed in the left column of Table 1. Then, we find the estimated integration order of series with Dickey-Pantula test and Bridge for 10 000 replications. These are also reported in Table 1. Results show that both estimators perform very well. There is no size distortion for $m_1 = m_2 = m_3 = 1$, $m_1 = m_2 = 1, m_3 = 0$, and $m_1 = 1, m_2 = m_3 = 0$. However, Bridge estimator has some size distortion for the set of roots 1.0, 0.7, 0.7 and 0.7, 0.7, 0.7. In these cases, Bridge estimator frequently concludes the integration order as 2 and 1, while the true order is 1 and 0, respectively. This means that Bridge estimator may lead to overdifference in some cases. It is also observed that the Dickey-Pantula test underdifference the model more than the Bridge estimator does. When we compare the correct model selection percentages for the remaining set of roots, we observe that Bridge dominates Dickey-Pantula test in 6 cases while Dickey-Pantula test dominates Bridge in 6 cases as well. It is observed that Bridge outperforms Dickey-Pantula test especially when the roots become closer to 1.

Tab. 1: Model selection percentages for 10 000 replications

Selected Model Roots	$d = 3$	$d = 2$	$d = 1$	$d = 0$
1.0, 1.0, 1.0	95.09*	4.91	0.00	0.00
	94.99*	4.74	0.24	0.03
1.0, 1.0, 0.7	1.70	**98.22***	0.08	0.00
	0.00	94.76*	4.89	0.35
1.0, 1.0, 0.5	0.08	**99.18***	0.74	0.00
	0.00	94.88*	4.81	0.31
1.0, 1.0, 0.0	0.00	94.48*	5.52	0.00
	0.00	**94.77***	4.96	0.27
1.0, 0.7, 0.7	0.00	51.53	48.42*	0.05
	0.00	2.68	**91.71***	5.61
1.0, 0.7, 0.5	0.00	7.07	92.49*	0.44
	0.00	0.31	**94.18***	5.51
1.0, 0.7, 0.0	0.00	0.04	**97.31***	2.65
	0.00	0.02	94.63*	5.35
1.0, 0.5, 0.5	0.00	0.08	**97.93***	1.99
	0.00	0.02	94.56*	5.42
1.0, 0.5, 0.0	0.00	0.00	**95.80***	4.20
	0.00	0.00	94.63*	5.37
1.0, 0.0, 0.0	0.00	0.00	94.42*	5.58
	0.00	0.00	**94.52***	5.48
0.7, 0.7, 0.7	0.00	0.12	49.41	50.47*
	0.00	0.00	17.66	**82.34***
0.7, 0.7, 0.5	0.00	0.02	17.60	82.38*
	0.00	0.00	9.33	**90.67***
0.7, 0.7, 0.0	0.00	0.00	3.92	**96.08***
	0.00	0.00	4.15	95.85*
0.7, 0.5, 0.0	0.00	0.00	1.08	98.92*
	0.00	0.00	0.73	**99.27***
0.5, 0.5, 0.5	0.00	0.00	0.58	99.42*
	0.00	0.00	0.55	**99.45***

Note: The top line on each set represents the Bridge estimator; the bottom line represents the Dickey-Pantula test. (*) indicates the true model and bold cases indicates the outperforming method for the selected set of roots.

Conclusion

In this paper we suggested Bridge estimator for testing multiple unit roots. We compared our proposed estimator with the existing Dickey-Pantula test. Our simulations showed that none of these methods is uniformly better than the other. However, results suggested that Dickey-Pantula test underdifference the model more than Bridge estimator does. We also found that Bridge estimator outperforms Dickey-Pantula test, especially when the roots become closer to one. For the future research, one may find the limiting distribution of Bridge estimator under multiple unit roots. It is also possible to extend our model by including lags and deterministic components. We expect that this might increase the performance of Bridge estimator since it is a model selection procedure and eliminates the two-step nature of unit root tests.

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