ON-LINE CALIBRATION OF THE EWMA MODELS: SIMULATIONS AND APPLICATIONS

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Abstract

The exponentially weighted moving average (EWMA) model is a particular modelling scheme used by RiskMetrics for predicting the current level of volatility of financial time series. It is designed to track changes in volatility by assigning exponentially decreasing weights to the observed historical squared financial returns. The applied weighting factors are conventionally prescribed by experts (users), or they are estimated employing standard statistical inference procedures, e.g. the maximum likelihood method. However, it is also possible to consider recursive (sequential or on-line) estimation techniques, which represent numerically effective alternatives to the already established approaches. The aim of this paper is to introduce and study a one-stage self-weighted on-line estimation algorithm appropriate for calibrating the EWMA model. Firstly, its derivation and theoretical properties are briefly outlined and summarized. Secondly, its practical performance is investigated by various Monte Carlo simulations. Lastly, the suggested calibration scheme is examined in the context of empirical financial data. In particular, volatility of the central index of Prague Stock Exchange (PX index) is monitored using the suggested estimation technique to reflect (eventual) structural changes of the EWMA parameter.

Key words: EWMA model, PX index, recursive estimation, RiskMetrics, simulations

JEL Code: C01, C51, C58

Introduction

The exponentially weighted moving average (EWMA) model is a particular conditional heteroskedasticity modelling scheme. This theoretical approach is frequently linked to investigating financial time series, more specifically to monitoring time-varying volatility. The EWMA model has been primarily developed as an alternative to the GARCH model for anticipating future volatility of financial returns. The name of this concept originates from the fact that the conditional variance is an exponentially weighted sum of historical squared financial returns with the geometrically declining weights going back in time (Tsay, 2013). Therefore, this model is simply capable to track changes in volatility. Since its introduction, it has been investigated from various theoretical and practical perspectives (Morgan, 1996). It has been successfully applied in many empirical studies. Moreover, the EWMA framework is regarded as the benchmark by many practitioners. An extensive body of academically and practically oriented literature exists in this field of research. For example, one may employ the EWMA model to improve the quality of predicted volatility traded on future markets (Covrig & Low, 2003), to examine volatility of various stock indices (Walsh & Tsou, 1998), or to calculate distinct risk measures (e.g. Value at Risk) for constructing an optimal portfolio of risky financial assets (Brooks & Persand, 2003).

The value of the only parameter of the EWMA model defining the discussed geometrically declining weights is usually prescribed by experts or users (Morgan, 1996). Alternatively, it can be calibrated employing standard (off-line) statistical inference procedures (e.g. the conditional maximum likelihood method). However, it is indeed rarely estimated sequentially (Tsay, 2013). On the other hand, it might be advantageous to adopt a numerically effective technique that could estimate and control this parameter (or model) in real time. For instance, one can apply this approach in the case of high-frequency data.

The aim of this contribution is to introduce and study a one-stage self-weighted recursive estimation method for calibrating the EWMA model on-line. The suggested algorithm has been derived using standard recursive identification instruments (Ljung, 1999). This method has demonstrated its numerical capabilities by means of simulations and an empirical application.

This paper is organized as follows. Section 1 reviews the EWMA modelling framework jointly with its fundamental features and introduces corresponding off-line estimation procedures. Section 2 derives and briefly comments the one-stage self-weighted recursive algorithm for calibrating the EWMA model. Section 3 analyzes this estimation method by Monte Carlo experiments. Section 4 considers an empirical application of this methodology. The paper is summarized by conclusions.

1 EWMA model

Formally, the EWMA model of returns $\{y_t\}$ is commonly defined as (Morgan, 1996):

$$y_t = \sigma_t \varepsilon_t, \quad \sigma_t^2 = (1-\lambda) y_{t-1}^2 + \lambda \sigma_{t-1}^2, \tag{1}$$

where the only modelling parameter λ lies in the interval (0,1) and $\{\varepsilon_t\}$ is a sequence of i. i. d. random variables with zero means and unit variances. One can compute the following conditional moments:

$$\mathrm{E}(y_t \mid \Omega_{t-1}) = 0, \quad \mathrm{var}(y_t \mid \Omega_{t-1}) = \mathrm{E}(y_t^2 \mid \Omega_{t-1}) = \sigma_t^2, \tag{2}$$

where Ω_t denotes the smallest σ-algebra with respect to which y_s is measurable for all $s \leq t$. Apparently, positivity of the conditional variance σ_t^2 is ensured by construction; see (1). The one-step-ahead prediction of σ_t^2 is expressed as:

$$\sigma_{t+1|t}^2 := \mathrm{E}(\sigma_{t+1}^2 \mid \Omega_t) = (1-\lambda) y_t^2 + \lambda \sigma_t^2 = \sigma_{t+1}^2. \tag{3}$$

Similarly, the k-step-ahead prediction of σ_t^2 is given by (for $k > 1$):

$$\sigma_{t+k|t}^2 := \mathrm{E}(\sigma_{t+k}^2 \mid \Omega_t) = \sigma_{t+1|t}^2. \tag{4}$$

To calibrate the EWMA model (1) using T observations $\{y_1, \ldots, y_T\}$, we usually employ one of these methods: (i) the value of λ is prescribed by experts (e.g. the choice 0.94 is obviously recommended for daily data); (ii) λ is estimated minimizing the root mean squared error of the forecast inaccuracies (y_t^2 - σ_t^2) assuming that y_0 and σ_0 are either defined or observed; (iii) supposing certain probability distribution of $\{\varepsilon_t\}$ (the Gaussian innovations are preferred in regarding to consistency of estimates), one may calibrate the parameter λ maximizing the conditional log-likelihood function (y_0 and σ_0 are known).

2 On-line calibration of the EWMA model

In this section, we shall introduce the one-stage self-weighted recursive estimation algorithm that can calibrate the parameter of the EWMA model (1) in real time. In many instances, this approach may be truly advantageous. For example, it is possible to monitor or predict volatility sequentially in the high-frequency financial data context. Recursive estimation methods are also effective in terms of memory storage and computational complexity since the current parameter estimates are evaluated using the previous estimates and actual measurements. Incidentally, they can be used to detect structural modelling changes.

Applying general recursive prediction error method (Ljung, 1999), one can derive the recursive scheme for on-line estimating the parameter λ of the EWMA model (1). Notice that the conditional log-likelihood criterion with Gaussian innovations $\{\varepsilon_t\}$ is assumed. This sequential algorithm can be concisely formulated as follows:

$$
\begin{aligned}
\hat{\lambda}_t &= \hat{\lambda}_{t-1} + \frac{\hat{p}_{t-1}(y_t^2 - \hat{\sigma}_t^2)\hat{\sigma}_t^{2}{}'}{\alpha_t(\hat{\sigma}_t^2)^2 + \left(\hat{\sigma}_t^{2}{}'\right)^2 \hat{p}_{t-1}}, \\
\hat{p}_t &= \frac{1}{\alpha_t}\left\{\hat{p}_{t-1} - \frac{\hat{p}_{t-1}^2\left(\hat{\sigma}_t^{2}{}'\right)^2}{\alpha_t(\hat{\sigma}_t^2)^2 + \left(\hat{\sigma}_t^{2}{}'\right)^2 \hat{p}_{t-1}}\right\}, \\
\hat{\sigma}_{t+1}^2 &= (1-\hat{\lambda}_t)y_t^2 + \hat{\lambda}_t\hat{\sigma}_t^2, \\
\hat{\sigma}_{t+1}^2{}' &= -y_t^2 + \hat{\sigma}_t^2 + \hat{\lambda}_t\hat{\sigma}_t^{2}{}', \quad t = 1,2,...,
\end{aligned}
\tag{5}
$$

where $\hat{\lambda}_t$ denotes the recursive estimate of the parameter λ. We recommend initializing the previous procedure under these conditions (Ljung, 1999): (i) $\hat{p}_0$ is a large positive number, e.g. $\hat{p}_0 = 10^5$; (ii) $\hat{\lambda}_0$ should be taken from the interval (0,1), e.g. as 0.94 as is usually preferred by RiskMetrics for daily data; (iii) $\hat{\sigma}_1^2$ is a positive number (e.g. the sample variance of several first measurements) and $\hat{\sigma}_1^2{}'$ equals zero; (iv) $\{\alpha_t\}$ is a deterministic sequence of positive real numbers smaller or equal to one that either accelerates convergence or allows tracking parameter changes (see the discussion below and also Section 3).

At each time t, it is necessary to check whether the recursive estimate belongs to the interval (0,1) before evaluating other quantities in (5). If not, one should artificially set the current estimate as the previous one to avoid eventual specification problems. This simple projection ensures positivity of the conditional variance. The sequence $\{\alpha_t\}$, the so-called forgetting factor, may be selected as follows: (i) α_t gradually grows to 1 as t goes to infinity, e.g. $\alpha_t = 0.99\alpha_{t-1} + 0.01$, $\alpha_0 = 0.95$; (ii) $\alpha_t = \alpha$ for some α in (0,1), e.g. $\alpha = 0.995$, and all t. The first option corresponds to estimating the model (1) supposing time-invariant λ. The increasing forgetting factor improves the convergence speed of the algorithm during the transient phase. The second one is associated with the eventuality that λ can vary over time. The constant forgetting factor less than one progressively reduces the influence of historical measurements, and thus enables to track parameter changes.

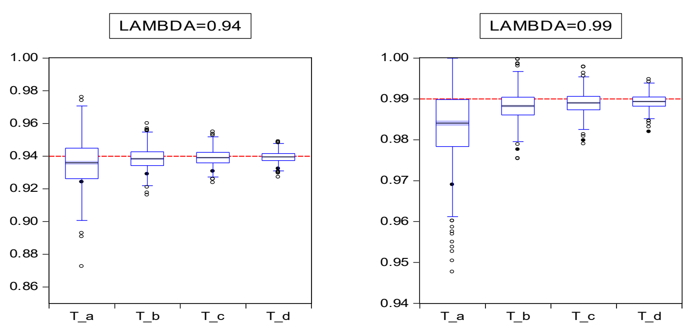
Theoretical properties of the suggested recursive estimation algorithm coincide with the off-line case (as t goes to infinity), where the corresponding conditional log-likelihood criterion is maximized. Namely, convergence and asymptotic distributional characteristics are identical for a sufficiently large portfolio of observations (Ljung, & Söderström, 1987).

3 Monte Carlo experiments

This section briefly investigates the proposed recursive estimation technique by means of Monte Carlo simulations. Various numerical experiments have been performed with almost analogical results. Therefore, only two representative instances are reviewed here. Particularly, we replicated two EWMA processes (1) with Gaussian disturbances of the length 10000 with two distinct parameters λ (i.e. 0.94 and 0.99) in order to study convergence properties of the suggested estimation method. We generated one thousand repetitions. The chosen length corresponds to an approximately three-hour dataset working with one-second data. All computations were conducted in the statistical software **R**.

Figure 1 illustrates numerical behaviour of the one-stage self-weighted recursive prediction error procedure defined by (5) specified by the consequent recommendations (α_t gradually grows to one as before). The estimation process was stopped at the times T_a = 1000, T_b = 3000, T_c = 5000, and T_d = 10000; the current estimates were always stored. Figure 1 summarizes sample characteristics of these estimates using the box-plots for each stopping time. It is apparent that the estimates converge to the true values jointly with decreasing variances. Thus, one might conclude that the suggested self-weighted recursive method (5) is capable to estimate the EWMA parameter in accordance with the literature (Ljung, 1999).

Fig. 1: Results of Monte Carlo simulations for both analysed EWMA processes



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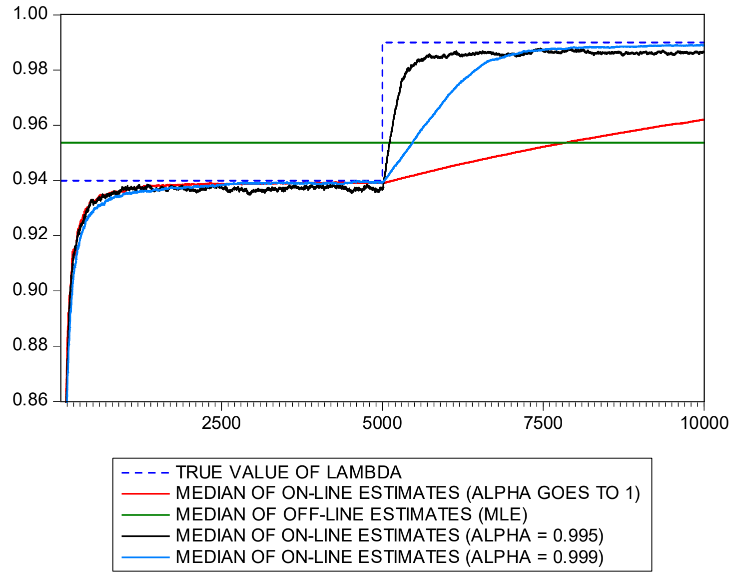
Additionally, one may examine the empirical behaviour of different estimating procedures for the case of the EWMA model (1) with the parameter varying over time and compare on-line and off-line methods altogether. Namely, we replicated two EWMA processes (1) with Gaussian innovations of the length 10000 with the time-varying parameter λ (specifically, λ = 0.94 for t = 1, ..., 5000, and λ = 0.99 for t = 5001, ..., 10000). One thousand replications were generated. Then, we analyzed convergence properties of these estimation procedures: (i) the recursive estimation scheme (5) with α_t increasing to one given as above; (ii) the recursive estimation scheme (5) with α_t = 0.995 for all t; (iii) the recursive estimation scheme (5) with α_t = 0.999 for all t; finally (iv) the usual (off-line) conditional maximum likelihood method suitable for the normally distributed EWMA model (Tsay, 2013).

Figure 2 draws these four estimates in detail. To be more precise, medians of available estimates have been computed at each time t, and they are represented in Figure 2. Clearly, the off-line (conditional maximum likelihood) estimates have demonstrated the least accuracy. The self-weighted recursive algorithm (5) with the forgetting factor growing to one has slowly reflected the underlying change in the parameter. Finally, the on-line calibration procedures with the constant forgetting factors have been able to track the change more conveniently. The closer the forgetting factor is to one, the estimates are more conservative, i.e. less volatile but also less precise. To conclude, the recursive estimation scheme (5) with the particular choices of $\{\alpha_t\}$ has outperformed the off-line estimator in this case. The off-line method is not adapted to follow the changes in the EWMA parameter over time.

Fig. 2: Comparison of medians of the various EWMA parameter estimates



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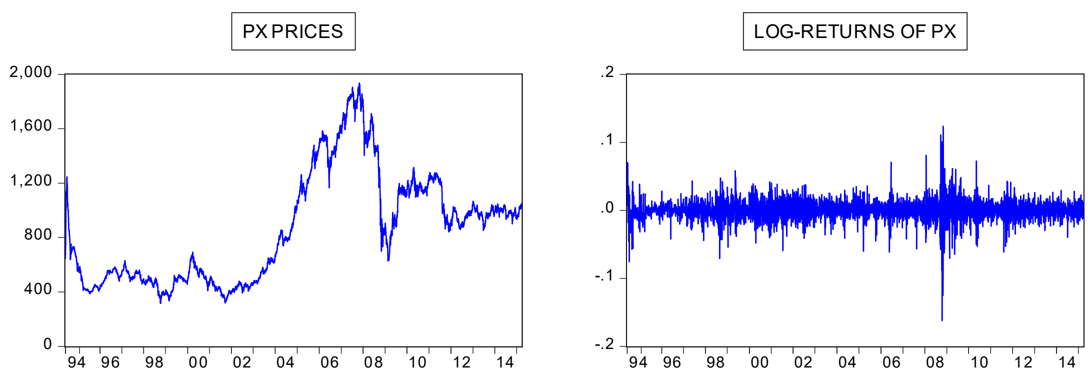
4 Empirical application: The PX index

The PX index (ISIN XC0009698371) is an official market-cap weighted stock index composed of the most liquid shares traded on the Prague Stock Exchange. In particular, it is a price index of blue chips issues, which is calculated in real-time and weighted by market capitalization. Dividends are not considered. A new value of the PX index is delivered by a particular formula; it reflects each single price change of index constituents. The maximum weight for a share issue is 20% on a decisive day. A portfolio of basic issues is variable, and it can be restructured quarterly (Wiener Borse, 2015).

The PX index was launched on 5th April 1994 (initially known as PX-50). Its base was composed of the fifty most significant share issues operating on the Prague Stock Exchange. The opening base value was fixed on 1000. The number of core issues has been variable since December 2001. In March 2006, the PX index was officially introduced. It took over the whole history of the replaced index PX-50 continuing in its development. In March 2015, the PX base contained fourteen issues. The top five stocks had approximately 85% share of market capitalization in the portfolio. The majority of capitalization was allocated in banking, energy, and insurance sectors. Further details (including historical data) can be found on the official web pages of the Prague Stock Exchange (Prague Stock Exchange, 2015).

Figure 3 presents all historical daily closing quotes of the PX index until 31st March 2015 (i.e. 5248 observations). The minimal value 316 occurred on 8th October 1998 after the Russian financial crisis. The maximal observation 1936 was achieved 29th October 2007. It is visible that the crisis year 2008 was truly exceptional (Hendrych, 2014).

Fig. 3: Historical closing quotes and log-returns of the PX index



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In this section, we accept the pragmatic argument that it may be convenient to study (long) time series of daily financial (logarithmic) returns by conditional heteroskedasticity models assuming time-varying parameters (Trešl, 2011). Under this supposition, we shall analyze the daily log-returns of the PX index closing quotes employing the EWMA model calibrated by the suggested estimation scheme (5) with the constant forgetting factors.

In particular, the observed daily returns are investigated by the following methods: (i) the estimation scheme (5) using α_t that grows to one (see above); (ii) the algorithm (5) with $\alpha_t = 0.995$ for all t; (iii) the algorithm (5) with $\alpha_t = 0.997$ for all t; finally (iv) the off-line conditional maximum likelihood procedure. The on-line estimates delivered applying (i)-(iii) can track time-varying parameters as can be concluded from Figure 2.

Figure 4 presents the different paths of the estimated EWMA parameter λ for the PX index log-returns. Apparently, the recursive estimates calculated by (5) using the constant forgetting factors fluctuate around the off-line one very similarly. One can discover similar trends, which obviously correspond to the overall development of the PX index historical closing quotes (compare with Figure 3). However, all mentioned recursive estimates are less reliable at the beginning of the observed time series since these methods must be properly initialized at first (they are more sensitive here).

Fig. 4: Different estimates of the EWMA parameter (the case of the PX log-returns)

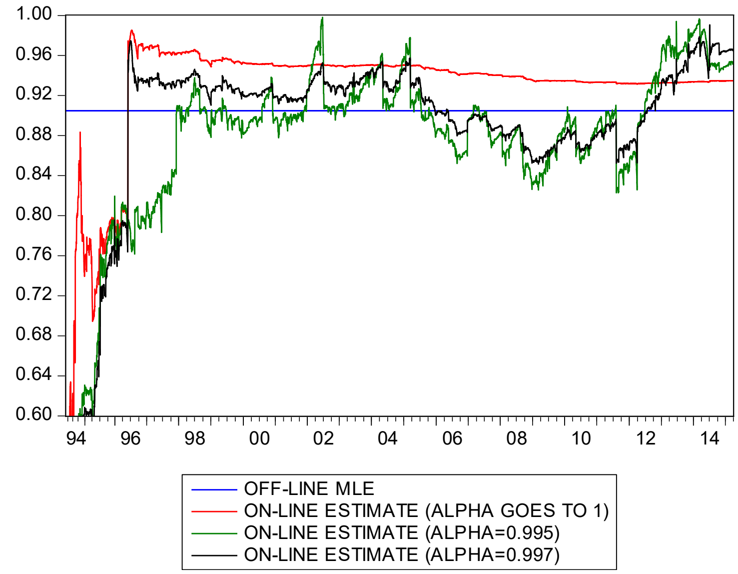

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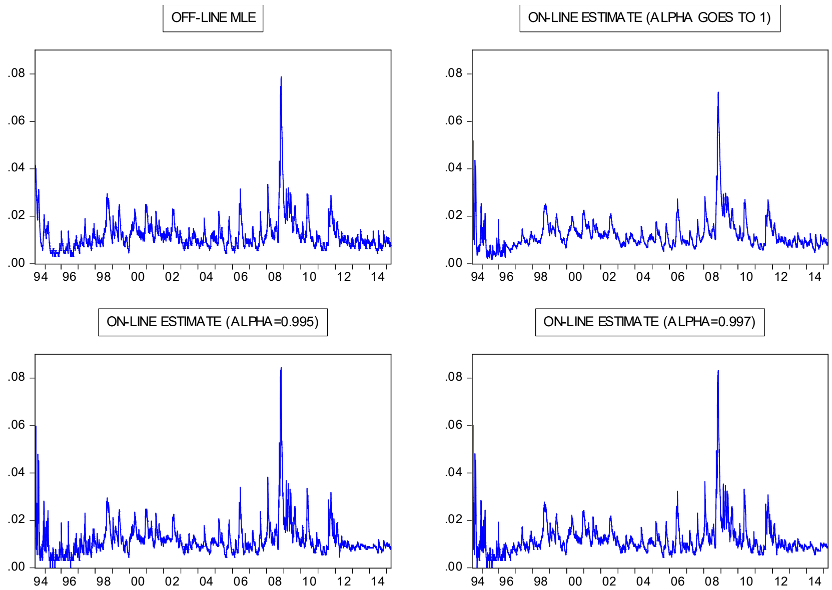
Figure 5 investigates the calculated conditional volatilities of the logarithmic returns of the PX index employing all previous procedures for calibrating the EWMA parameter λ. At first sight, one could conclude that all considered outputs follow analogical trends. The estimates based on the off-line conditional likelihood procedure and the suggested recursive algorithm (5) seem to be closely related. Nonetheless, to realize which of the studied calibration techniques offer the better predictions of volatility of financial returns, it would be necessary to employ additional criteria that question this issue from the financial management point of view (Patton, 2011). Alternatively, one may compare the achieved values of the (conditional) log-likelihood function associated with the particular estimation problem (or equivalently contrast some information criteria). These are recapitulated in Table 1. The complete and truncated samples are considered in calculating the presented log-likelihoods. Namely, first 10%, 30%, and 50% observations were cut off to verify the adequacy since the recursive methods are less stable during the initial phase of estimation (see above). Consequently, the estimation algorithm (5) introduced in Section 2 is visibly competitive.

Fig. 5: Conditional volatilities of the PX log-returns using various calibration methods



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Tab. 1: Values of the log-likelihood functions associated with the estimates in Figure 5

Sample | Method	*Off-line*	*On-line ($\alpha_t \to 1$)*	*On-line ($\alpha_t = .995$)*	*On-line ($\alpha_t = .997$)*
Complete	**15819.37**	15743.82	15748.40	15745.98
Truncated (first 10%)	14177.85	14155.86	14176.99	**14178.32**
Truncated (first 30%)	11005.22	10989.78	11010.51	**11011.65**
Truncated (first 50%)	7859.44	7840.45	**7865.21**	7863.89

Source: Author

Conclusion

In this paper, we introduced the one-stage self-weighted recursive estimation algorithm for calibrating the RiskMetrics EWMA model employing the general recursive identification instruments. The qualities of the proposed method have been demonstrated using simulations. The accepted modelling framework has been applied to investigate volatility of the PX index comparing different estimators. It has proved its competitiveness and usefulness. These findings motivate further research on on-line estimation and its possible practical applications (e.g. in the option pricing or optimal portfolio framework).

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