MORTALITY AT HIGHER AGES – COMPARISON OF THE GOMPERTZ-MAKEHAM FUNCTION AND ITS MODIFICATION

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Abstract

When we examine mortality at higher ages, we run into problems with small numbers of living and deaths. Lengthening of human life cause a growth in these numbers, but they are still too small (comparing with lower ages). These data are very often influenced by systematic and random errors. That is why the age-specific death rates are modelled by different analytical functions. Among the most famous and very often used functions could be included standard Gompertz-Makeham function. But it is not very good for capturing mortality at the highest ages that is why the modified Gompertz-Makeham function was created. In this paper an attention will be aimed to the Czech and Swedish population and their mortality in selected years (mortality in higher ages will be modelled by Gompertz-Makeham function and its modification).

Key words: mortality at the highest ages, Gompertz-Makeham function, modified Gompertz-Makeham function

JEL Code: J10, J11, J19

Introduction

During the last days changes in mortality especially in the highest ages are more and more important not only for demographers (Arltová et al., 2013 or Kannisto et al., 1994). But it is influenced by systematic and random errors. That is why it is good to model it. For modelling of mortality here will be used the Gompertz-Makeham function and its modification.

1 Methodology

As was mentioned before, for modelling of mortality in higher ages could be used the Gompertz-Makeham function (Gompertz, 1825, Makeham, 1860 or Thatcher, 1999). It is suitable approximately until 85 years (mortality is overestimated for higher ages) (Thatcher et al., 1998).

Gompertz-Makeham function is given by the formula: $\mu_x = a + b.c^x$, (1)

where x is age a, b and c are parameters of the Gompertz-Makeham function, μ_x is the force of mortality.

The Gompertz-Makeham function is based on the assumption of constant relative increase in the force of mortality with increasing age (but it is not true in the highest ages) (Dotlačilová, 2013, Ekonomow, Yarigin, 1989). That is why the modified Gompertz-Makeham function was created. It is based on the assumption of decreasing relative increase in the intensity of mortality with an increasing age. It is suitable for ages 85 and higher. But it is not appropriate at lower ages (Burcin et al., 2010).

The formula of modified Gompertz-Makeham function: $\mu_x = a + b.c^{x_0 + \frac{1}{d}.\ln[(x - x_0)+1]}$, (2)

where x_0 is age from which smoothing of mortality curve will be performed by modified Gompertz-Makeham function a, b and c are parameters of the Gompertz-Makeham function, d is parameter of modified Gompertz-Makeham function.

Before the estimation of parameters of the Gompertz-Makeham function is good to mention the relationship between the age-specific death rate and the force of mortality (Dotlačilová, Langhamrová, 2014):

$$m_x \doteq \mu(x + 0{,}5) \tag{3}$$

For the estimation of unknown parameters of the Gompertz-Makeham function could be used the King-Hardy methodology. The first step is based on the observation that the Gompertz-Makeham function has 3 unknown parameters and for their estimation we need 3 equations. Initial estimation formulas are derived from these equations (Šimpach, 2012):

$$G_1 = \sum_{x=x_0}^{x_0+k-1} m_x = \sum_{x=x_0}^{x_0+k-1} (a + b.c^{x+0,5}) = k.a + b.c^{x_0+0,5}.(1 + c + \ldots + c^{k-1}), \tag{4}$$

$$G_2 = \sum_{x=x_0+k}^{x_0+2k-1} m_x = k.a + b.c^{x_0+k+0,5}.(1 + c + \ldots + c^{k-1}), \tag{5}$$

$$G_3 = \sum_{x=x_0+2k}^{x_0+3k-1} m_x = k.a + b.c^{x_0+0,5+2k}.(1 + c + \ldots c^{k-1}), \tag{6}$$

where x_0 is the initial age, from which sums of empirical values of age-specific death rates are performed and k is widths of age interval. For own calculations we will consider x_0 = 60 a k = 8.

If we solve the equations at first we subtract them. We subtract equation (5) from equation (6). Afterthat we subtract equation (4) from equation (5).

$$G_3 - G_2 = b.c^{x_0+k+0,5}.(c^k - 1).(1 + c + ... + c^{k-1}),$$

$$G_2 - G_1 = b.c^{x_0+0,5}.(c^k - 1).(1 + c + ... + c^{k-1}).$$

We gradually exclude parameters a and b. It remains only parameter c power k (c^k):

$$c^k = \frac{G_3 - G_2}{G_2 - G_1} = \frac{b.c^{x_0+k+0,5}.(c^k - 1).(1 + c + ... + c^{k-1})}{b.c^{x_0+0,5}.(c^k - 1).(1 + c + ... + c^{k-1})}. \tag{7}$$

Unknown parameter c we obtain like k-root from c^k.

Death rates probably will be increasing in the highest ages. So it is important to check if it is true:

$$b \neq 0,\ c \neq 0,\ c \neq \pm 1. \tag{8}$$

Ed. 1 From the equation (1) it is clear that if parameter b or parameter c will be equal to 0 death rates will be constant (independent on age). But it is not true in the highest ages.

After further adjustment we obtain the initial estimates for parameters b and a like:

$$b = \frac{G_2 - G_1}{c^{x_0+0,5}.(c^k - 1).(1 + c + ...c^{k-1})}, \tag{9}$$

$$a = \frac{G_1 - b.c^{x_0+0,5}.(1 + c + ... + c^{k-1})}{k}. \tag{10}$$

For parameter d (parameter of modified Gompertz-Makeham function) does not exist initial estimation formula. Therefore it is firstly selected its initial value (d = 0,02), which will be optimized after that.

For initial estimates of parameters of the Gompertz-Makeham function it is possible to perform their subsequent optimization (for the improvement of obtained smoothing according to the Gompertz-Makeham function). Here we can use optimization criterion like minimization of sum of weighted squared deviations.

$$WSD = \frac{S_{t,x} + S_{t+1,x}}{2.m_{t,x}.(1 - m_{t,x})}.(m_{t,x} - \widetilde{m}_{t,x}^{GM})^2, \qquad x \in \langle 60; y \rangle \tag{11}$$

where $S_{t,x}$ is the number of mid-year population at age x and at the beginning of year t, $m_{t,x}$ are age-specific death rates in year t, $\widetilde{m}_{t,x}^{GM}$ are smoothed age-specific death rates obtain by the Gompertz-Makeham function and y is the highest age for which we have non-zero value of age-specific death rate.

In the next step is important to divide sum of weighted squared deviations into two parts: S_1 (it is connected with optimization of parameters a, b, c) and S_2 (it is connected with optimization of parameter d).

$$S_1 = \sum_{x=60}^{82} WSD. \qquad S_2 = \sum_{x=83}^{y} WSD. \tag{12}$$

2 Results

For own calculations was used data about mortality in the Czech Republic and in Sweden. The aim is modelling of mortality in higher ages by the Gompertz-Makeham function and after that use the combination of the Gompertz-Makeham function and its modification (modified Gompertz-Makeham function will be used from age 83). Based on the obtained results was done the evaluation for each decade.

For each graph the Gompertz-Makeham function is displayed by light grey colour and combination of the Gompertz-Makeham function and its modification by dark grey colour.

Results for males and for females in the Czech Republic are shown in the first two groups of graphs.

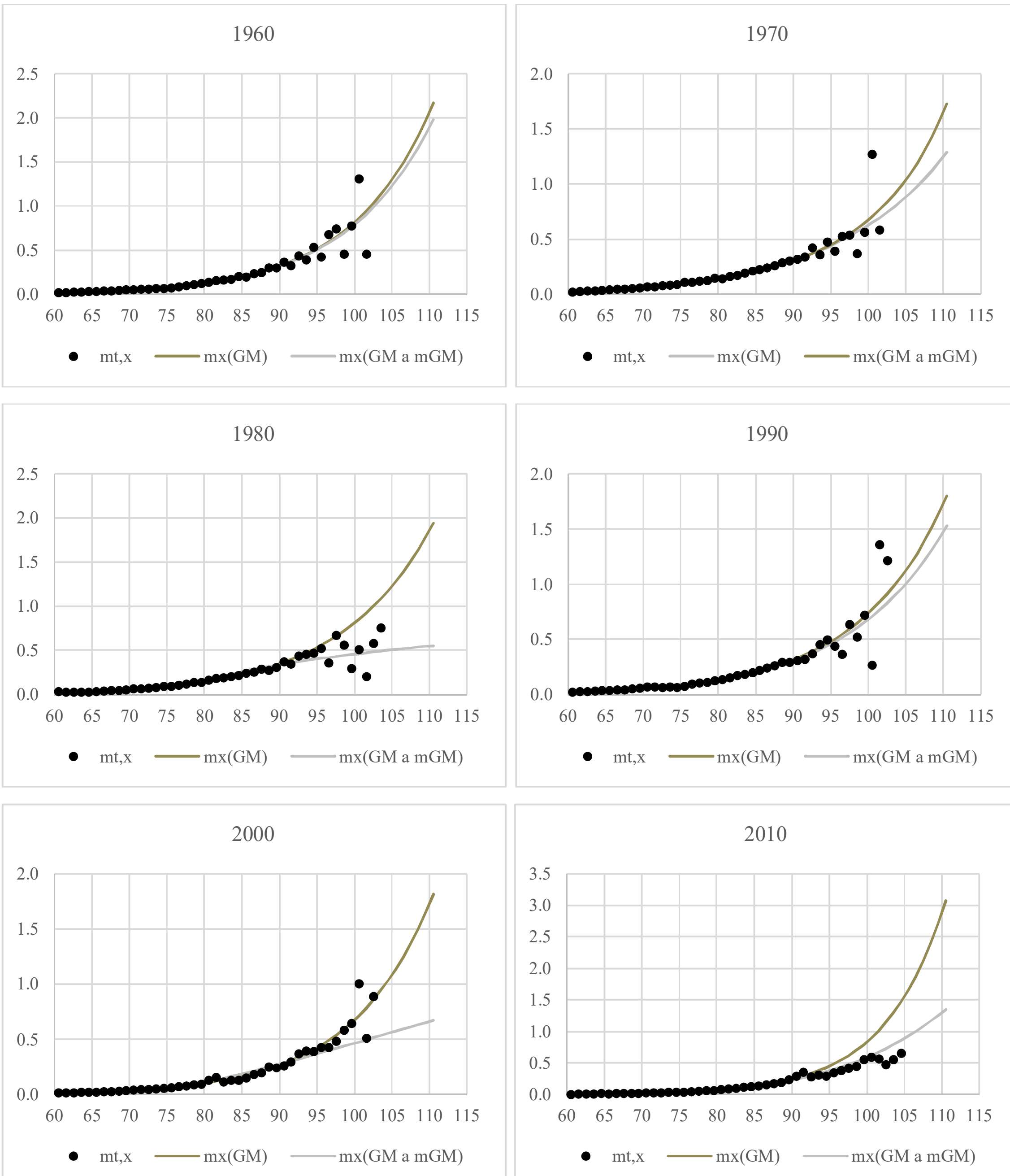
From obtained graphs for males in the Czech Republic it is clear that mortality gradually decreased not only in higher ages (it is obvious especially from higher concentration of empirical death rates in lower ages).

On the other hand we can see that the Gompertz-Makeham function and its modification are more and more different from the past to the present. We can also say that the combination of the Gompertz-Makeham function and its modification is more suitable at the present. The Gompertz-Makeham function more overestimates the empirical values of mortality since 1980.

The second group of graphs was made for females in the Czech Republic.

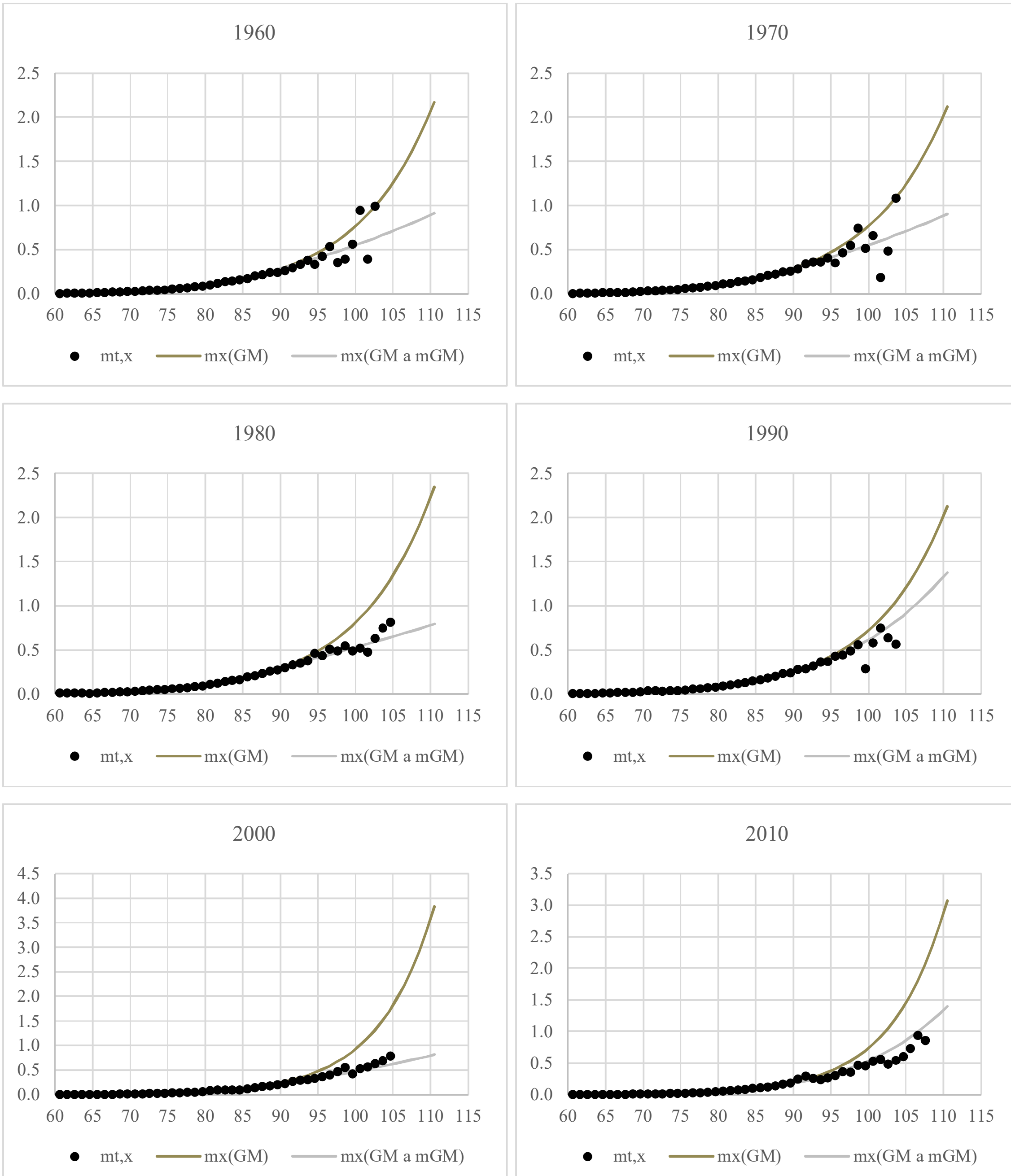
If we compare obtain smoothing we can see that obtained results are very different from the beginning of the reference period (especially for ages 90+). On the other hand it is clear that in every single decade it is more suitable to use combination of the Gompertz-Makeham function and its modification.

Fig. 1: Mortality of males in the Czech Republic



Source: data: Human Mortality Database (HMD, 2014), author's calculation

Fig. 2: Mortality of females in the Czech Republic

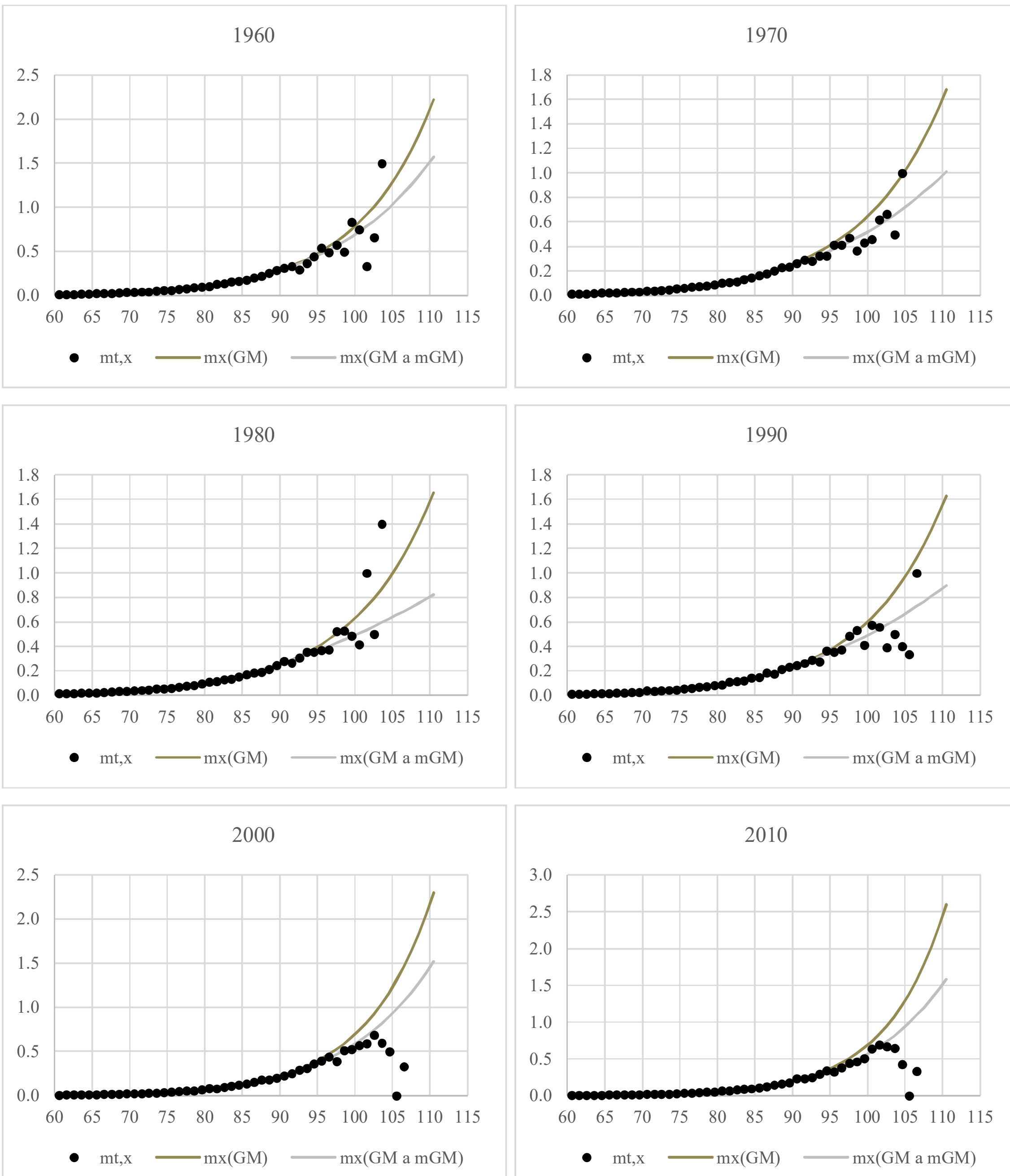


Source: data: Human Mortality Database (HMD, 2014), author's calculation

Mortality in Sweden is shown in the second two groups of graphs.

The first group shows mortality of Swedish males at the beginning of each decade. As for previous cases we can conclude that the combination of the Gompertz-Makeham function and its modification is better.

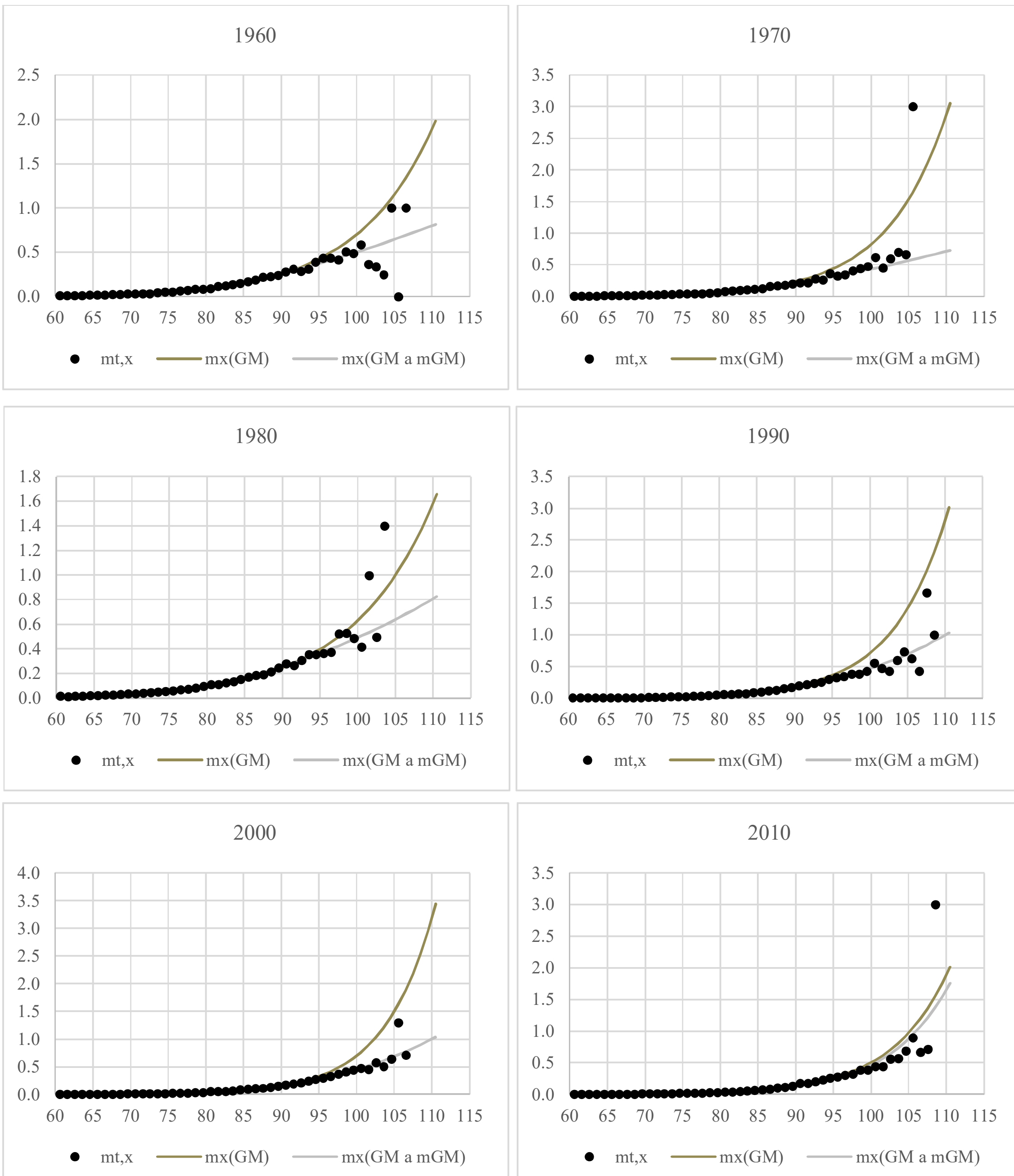
Fig. 3: Mortality of males in Sweden



Source: data: Human Mortality Database (HMD, 2014), author's calculations

The last group of graphs shows mortality of Swedish females in selected years. We can conclude that also here it is better to use the combination of the Gompertz-Makeham function and its modification.

Fig. 4: Mortality of females in Sweden



Source: data: Human Mortality Database (HMD, 2014), author's calculations

Conclusion

Base on the obtained results we can say that during the reference period we can observe an improvement in mortality for the Czech and Swedish population. We can also say that both smoothing methods are increasingly different from the past to the present (the exception is mortality of Swedish females in year 2010, where both methods give us similar curve).

We can also make a conclusion that for all populations is better to use combination of the Gompertz-Makeham function and its modification (except year 1960, when we obtained similar result for population of Czech males according to both methods).

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