MODELLING PAIRS OF EUROPEAN STOCK INDICES WITH COPULAS

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Abstract

This paper explores the copula approach for econometric modelling of joint parametric distribution. In this paper we have demonstrated our approach which uses bivariate time - varying EWMA - copula model. This approach is used to model the dynamic dependence between selected European stock indices. We propose the EWMA - dynamics for the correlation parameters. The time varying parameters will be modelled using exponential moving averages process. Our analysis is based on the weekly returns time series on the selected European stock indices. We have analyzed marginal distributions, and we have calibrated normal, Student's *t* - copula and Clayton copula with time - varying parameters and time - invariant parameters. As a marginal distribution we have taken into account normal and Student's *t* - distribution. These distributions are the best fitting distributions for our analyzed data. We have shown that for data with lower frequency it is sufficient to use our approach.

Key words: EWMA, dynamic copula, normal copula, *t*-copula

JEL Code: C10, C58, G15, G10

Introduction

The relationship between the financial instruments is an important factor in portfolio management. The analysis of the time series of returns shows that returns are not always normal. In this paper we present a methodology for modelling dependencies between pair market indices using dynamic copula functions. The copula approach is a useful method for deriving joint distributions given by marginal distributions, especially when the variables do not follow normal distribution (Trivedi, Zimmer, 2005), (Embrechts et al., 2001), etc. On the other hand, copulas can be used to define nonparametric measures of dependencies for pairs of random variables. For investors, it is important to know whether prices of different assets exhibit dependence, particularly in the tails of the joint distributions. These models typically assume that asset prices have a multivariate normal distribution, but Embrechts et al. (2001) argue that this assumption is frequently unsatisfactory because large changes are observed more frequently than is predicted under the normality assumption. Since deviations from normality, e.g., tail dependence in the distribution of asset prices, greatly increase computational difficulties of joint asset models, modelling based on a copula parameterized by non-normal marginals is an attractive alternative; see (Bouye et al. 2000), (Patton 2006), (Trivedi, Zimmer 2005).

In this paper we focus on time varying copula approach with normal and Student's *t* marginals. Obviously we use static parameters for describing appropriate dependence parameters for copulas. If we take into account elliptical copulas (Gaussian or *t*-copula) we need to estimate the correlation coefficient. Dias & Embrechts (2010) used Fisher's transformation and Tse & Tsui (2002) time-varying correlation dynamics. In both cases, there has been estimated and forecasted a correlation coefficient within the context of the assumed model. Alexander (2008b) wrote that it must be emphasized that there is no absolute "true" variance or covariance. What "true" is depends only on the statistical model. Even if we knew for certain that our model is a correct representation of the data generation process, we could never measure the true variance and covariance parameters exactly because pure variance and covariance are not traded in the market. It is obvious that the estimates of the true correlation coefficients (or variance and covariance) are subject to sampling error. That is, even when we use the same model to estimate a variance, our estimates will differ depending on the data used. Both changing the sample period and changing the frequency of observations will affect the covariance/correlation matrix estimate. In this paper we concentrate on the exponential moving average time series models for variance and covariance/correlation coefficient, focusing on the copula implementation of the approach and providing an explanation for their advantages and limitations.

This paper is organized as follows. In Section 1 we introduce dynamic copula approach and its estimation methodology. In Section 2 we describe the data and our results. Section 3 summarizes and concludes the paper.

1 Dynamic copula approach

Let F^R refer to the multivariate normal cdf F^Σ with correlation matrix Σ, where the means are 0 and the standard deviations are 1. Let F be the cdf of a univariate standard normal. The Gaussian copula is given as

$$C_G^\Sigma(u_1,u_2)=F^\Sigma\left(F^{-1}(u_1),F^{-1}(u_2)\right), \tag{1}$$

A straightforward generalization of the Gaussian copula is the Student's *t* copula, which has more realistic tail properties. It is derived from the multivariate Student's *t* distribution. It is defined as follows:

$$C^{St}_{\nu,\Sigma}(u_1,u_2)=t_{\nu,\Sigma}\left(t^{-1}_{\nu_1}(u_1),t^{-1}_{\nu_2}(u_2)\right), \quad (2)$$

where $t_{\nu,\Sigma}$ denote two-dimensional Student's *t*- distribution with ν degrees of freedom and correlation matrix Σ. t_{ν_i} is one-dimensional Student's *t* distribution with ν_i degrees of freedom, i = 1, 2.

Another class of copulas creates the copulas called the Archimedean class that include for example Gumbel, Clayton, Frank, and generalized Clayton copulas. This class of copula functions is defined in terms of copula generator $\phi(u)$, which is a continuous, convex, and strictly decreasing function defined on the interval $[0,1]$ that ranges from 0 to ∞. The copula is then defined as

$$C(u_1,u_2)=\phi^{-1}(\phi(u_1)+\phi(u_2)). \quad (3)$$

2-dimensional Clayton copula generator is

$$\phi_\theta(u)=\frac{1}{\theta}\left(u^{-\theta}-1\right) \quad (4)$$

and the copula is given by

$$C^{Cl}_\theta(u_1,u_2)=\left(u_1^{-\theta}+u_2^{-\theta}-1\right)^{\frac{-1}{\theta}}, \theta\geq 0. \quad (5)$$

See (McNeil et al., 2005), (Engle, 2009), (Trivedi, Zimmer, 2005).

Let us have a random sample of the pairs of random variables (financial instruments) (Z_1,Z_2). The conditional distribution of Z_i can be written as

$$F^\Sigma(z_i;\alpha_1,\alpha_2,\theta_i)=C\left(F_1(z_{1i};\alpha_1),F_2(z_{2i;}\alpha_2);\theta_i\right). \quad (6)$$

We assume that each F_i is absolutely continuous with density f_i, the vector α_i is a vector of the parameters of the marginal distribution functions and parameter θ_i parameterizes the copula family.

Assuming that copula C has the density c given by

$$c(u_1,u_2;\theta)=\frac{\partial^2 C(u_1,u_2;\theta)}{\partial u_1\partial u_2},\ (u_1,u_2)\in[0,1]^2 \quad (7)$$

the density of Z_i is given by:

$$f(z_i;\alpha_1,\alpha_2,\theta_i)=c\left(F_1(z_{1,i};\alpha_1),F_2(z_{2,i};\alpha_2);\theta_t\right)f_1(z_{1,i};\alpha_1)f_1(z_{2,i};\alpha_2). \quad (8)$$

The log-likelihood function of the model is given as

$$l(\alpha_1, \alpha_2, \theta_i) = \sum_{i=1}^{2} \left(\log c\left(F_1(z_{1,i}; \alpha_1), F_2(z_{2,i}; \alpha_2); \theta_t\right) + \sum_{k=1}^{2} \log f_k(z_{k,i}; \alpha_k) \right). \tag{9}$$

Numerical maximization of equation (9) yields the maximum likelihood estimates of the model (see (Dias, Embrechts, 2010), Alexander (2008a), etc.).

Due to Sklar's theorem we can separate the estimation of the marginal distributions from the estimation of the model for the dependence structure. We can independently maximize each term of the marginal log-likelihood functions (9). The estimation $\hat{\alpha}_i$ is obtained maximizing $l(\alpha_i)$:

$$l(\alpha_i) = \sum_{k=1}^{2} \log f_k(z_{k,i}; \alpha_k), \; i = 1, 2 \tag{10}$$

The final function to maximize is

$$l(\theta_i) = \sum_{i=1}^{2} \log c\left(F_1(z_{1,i}; \alpha_1), F_2(z_{2,i}; \alpha_2); \theta_t\right). \tag{11}$$

From (11) we have obtained the dependence parameter θ_i, see (Dias, Embrechts, 2010), (Alexander 2008a) or (Patton 2006), etc.

1.1 EWMA estimation of the copula parameters

In this paper we refer to two types of the 2-dimensional copulas: elliptical type (normal, Student’s *t*-copula) and Archimedean type (Clayton copula). For elliptical copulas we need to estimate the correlation coefficients and for the Clayton copula parameter θ.

Consider a set of time series of the *i*-th log returns $\{r_{i,t}\}_{t=1}^{T}$, $i = 1,2,\ldots, m$, calculated by the formula:

$$r_{i,t} = \frac{\ln P_{i,t}}{\ln P_{i,t-1}}, \tag{12}$$

where $P_{i,t}$ denotes the closing prices of the *i*-th financial instrument at time *t*.

We estimate our parameters from log returns for the whole sample and for “rolling” samples. “Rolling samples” are calculated on a fixed size data “window” that is rolled through time, each time period adding a new return and taking off the oldest return. The length of this window of data is the time interval over which we compute the parameter. Copula methodology does not use the Pearson correlation coefficient because it is not a concordance measure. Therefore for an indication of the direction of co-movement of two returns time series we use the basic measures of dependencies and measures of concordance (Embrechts et al., 2001). Concordance measures are completely determined by the underlying copula and independent on the marginal distributions unlike Pearson's correlation (Hurd et al. 2007). The most widely known concordance measures are Kendall's tau and Spearman's rho. Kendall's tau is a difference between the probability of the variables either rising or falling together (concordance) minus the probability of them moving in different directions (discordance). Spearman's rho measures the correlation between rank ordered data (Embrechts et al., 2001), (Hurd et al. 2007), (Alexander 2008a), etc. In this paper we will approximate the Kendall's tau and Spearman's correlation coefficient using exponentially weighted moving averages for expressing the dynamics of the correlation coefficient.

Exponentially weighted moving average can be defined for any time series of data $(x_1,\ldots,x_t)$. The exponentially weighted average of these observations is defined as: ((Cipra, 2008), (Alexander 2008a, 2008b), (Lu, Huang, Gerlach, 2010))

$$\hat{x}_t = \frac{x_1 + \lambda x_2 + \lambda^2 x_3 + \cdots + \lambda^{t-2} x_{t-2}}{1 + \lambda + \lambda^2 + \cdots + \lambda^{t-2}} \approx (1-\lambda)\sum_{k=0}^{\infty} \lambda^k x_{t-k} \tag{13}$$

where λ is a smoothing constant, $0 < \lambda < 1$. The formula (16) can be rewritten in the form of recursion as:

$$\hat{x}_t = (1-\lambda)x_t + \lambda \hat{x}_{t-1} \tag{14}$$

It follows from equation (14) that if we set λ closer to 1 we prefer the persistency of perturbations resulting from the actual data. Thus, high λ gives little reaction to actual market events, but great persistence in perturbation and low λ gives highly reactive perturbations that quickly disappear. Limitation of exponentially weighted moving average models is that the reaction and persistence parameters are not independent: the strength of reaction to market events is determined by 1 – λ, whilst the persistence of shocks is determined by λ (Alexander, 2008a, b).

2 Analysis of the European indices

The paper presents the result of the analysis concerning the dependencies between the weekly returns for three pairs of indices: EURO STOXX 50 – DAX, EURO STOXX 50 –CAC, DAX – CAC. We have chosen three major European indices.

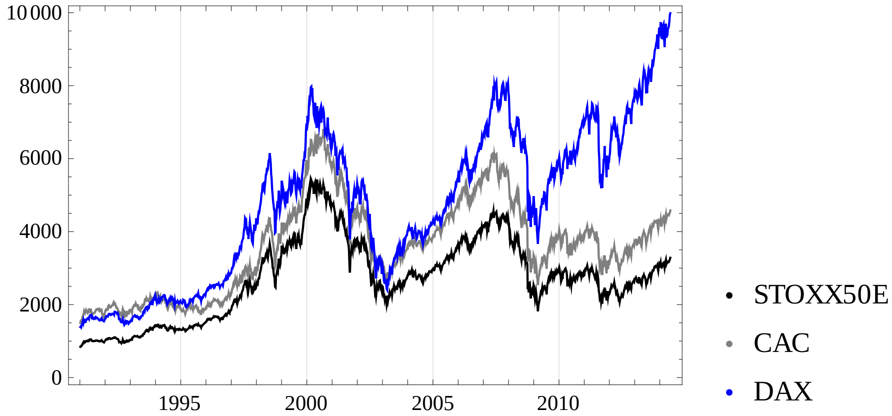
The EURO STOXX 50 Index (STOXX50E)[1] is licensed to financial institutions, and it provides a Blue-chip representation of supersector leaders in the Eurozone. German Stock

[1] http://www.stoxx.com/indices/index_information.html?symbol=SX5E

Index DAX 30 was formerly known as Deutscher Aktien IndeX 30[2]. The CAC 40 is a benchmark French stock market index[3]. All indices are denominated in Euro. The analyzed period has been from January 7, 1991 to June 2, 2014, (1222 observations) with weekly frequency (http://finance.yahoo.com).

Figure 1 shows time plots of the evolution of the European close price indices during analyzed period. It is evident that index DAX (blue line) was more sensitive in the known crisis periods as CAC index (grey line) or index EuroStoxx 50 (black line).

Fig. 1: Time plots of weekly quotations of EURO STOXX 50, CAC and DAX for the period January, 7, 1991-June, 2, 2014 , weekly frequency



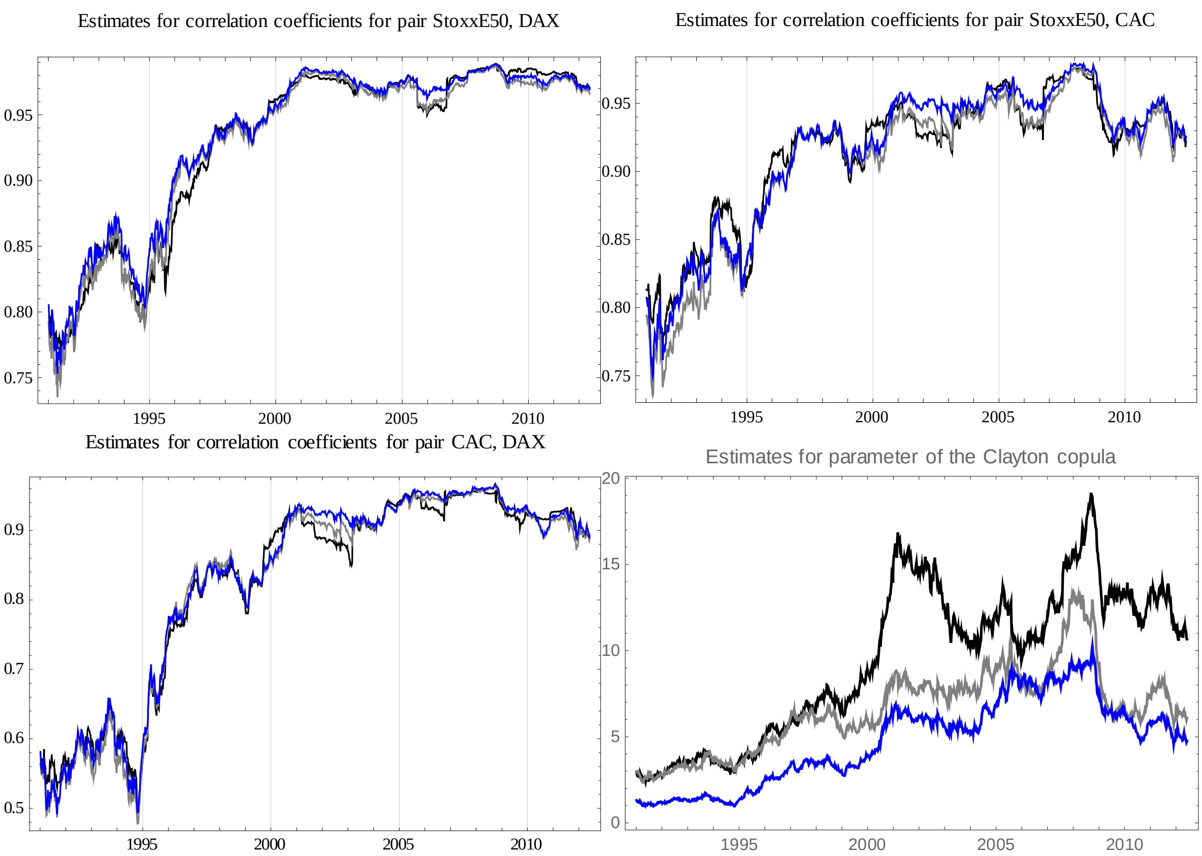
Source: Calculated by authors with Wolfram Mathematica software based on data from http://finance.yahoo.com

Unconditional pairwise correlation coefficients (Pearson's, Spearman's and Kendall's) estimation for all pairs of the European market indices log returns and for rolling windows (scope 2 years) from January 1991 to June 2014 are shown in figure 2. The plots enable easy determining of the periods of uncertainty in financial markets. An explosion of changes in the dependences visible in figure 2 is related to the crisis in 1996. The peaks are appearing in years 2002 and 2008. The current period can be characterized as a depression comparable with year 2004. The correlation coefficient has a range from 0.73 to 0.99 for pairs StoxxE50-DAX, StoxxE50-CAC respectively. The lower correlations characterize the pair DAX-CAC, the range is from 0.49 to 0.97. Long term and EWMA estimations with $\lambda = 0.97$ for all type of the correlation coefficients and θ parameter are shown in Table 1. We use high value of λ, and therefore the estimations of the correlation coefficients reflect to actual market events. Long term estimators of the correlations coefficients give correlation coefficients over the chosen sample. Pre-crisis periods are characterized by a decrease and subsequent increase in the relationship between market indices expressed by correlation coefficients. We see the

[2] http://www.moneycontrol.com/live-market/dax
[3] http://www.moneycontrol.com/live-market/cac

biggest decrease in correlation dependency in 1994; the significant changes appear in years 1998, 2002, 2007, 2012, too. If we compare the evolution of the correlation coefficients with evolution of θ parameter for Clayton copula over chosen sample (fig.2) we can see that Clayton copula parameter responds to higher fluctuations after the year 1999. Last extreme change was recorded in 2009. In general for period from 1991 to 1998 (the first two crises) the correlation coefficients are better suited for the graphical visualization of rapid changes. Crisis events after the year 2000 are better depicted by θ parameter for Clayton copula. The period after the 2000 was characterized by many economic problems in Greece, Ireland, Spain and therefore EU adopts the known economic policy.

Fig. 2: Correlation coefficient estimates for the indices returns StoxxE50, CAC, DAX, for the period January, 7, 1991-June, 2, 2014, weekly frequency (rolling window has 2 years) (Pearson's ρ–black line, Kendal's ρ–blue line, Spearman's ρ–gray line)



Source: Calculated by authors with Wolfram Mathematica software based on data from http://finance.yahoo.com

By analysing the data set with weekly frequency we have found that in the long-term data have Student's t - distribution with parameters listed in Table 2. But if we take into account the rolling window with length 104 weeks, 52 weeks or 26 weeks, we can find periods with the normal distribution; the degree of freedom of the Student's t - distribution is higher than 100. A change from the normal distribution to the Student's t - distribution only occurs when there is a sudden change in volatility, when the bull market changes to bear market and vice versa; it means during the crisis. Our analysed data with weekly frequency have persistent character in a local sense.

Tab. 1: Estimation of the Correlation coefficients

	Long term correlation			EWMA forecast of the correlation		
	Stoxx50-DAX	**Stoxx50-CAC**	**CAC-DAX**	**Stoxx50-DAX**	**Stoxx50-CAC**	**CAC-DAX**
Pearson's ρ	0.9431	0.9276	0.8539	0.9706	0.9222	0.8903
Kendal's ρ	0.9396	0.9225	0.8456	0.9694	0.9255	0.8932
Spearman's ρ	0.9333	0.9161	0.8349	0.9661	0.9248	0.8877
θ	6.9923	5.9285	3.5797	10.665	6.0872	4.7355

Source: Calculated by authors with Wolfram Mathematica software based on data from http://finance.yahoo.com

Tab. 2: Estimation of the parameters for Student's marginal, for the period January, 7, 1991-June, 2, 2014 , weekly frequency

	mean	Standard deviation	Degree of freedom
Stoxx50	0.0024	0.0215	4.627
DAX	0.0017	0.0238	5.888
CAC	0.0030	0.0231	4.471

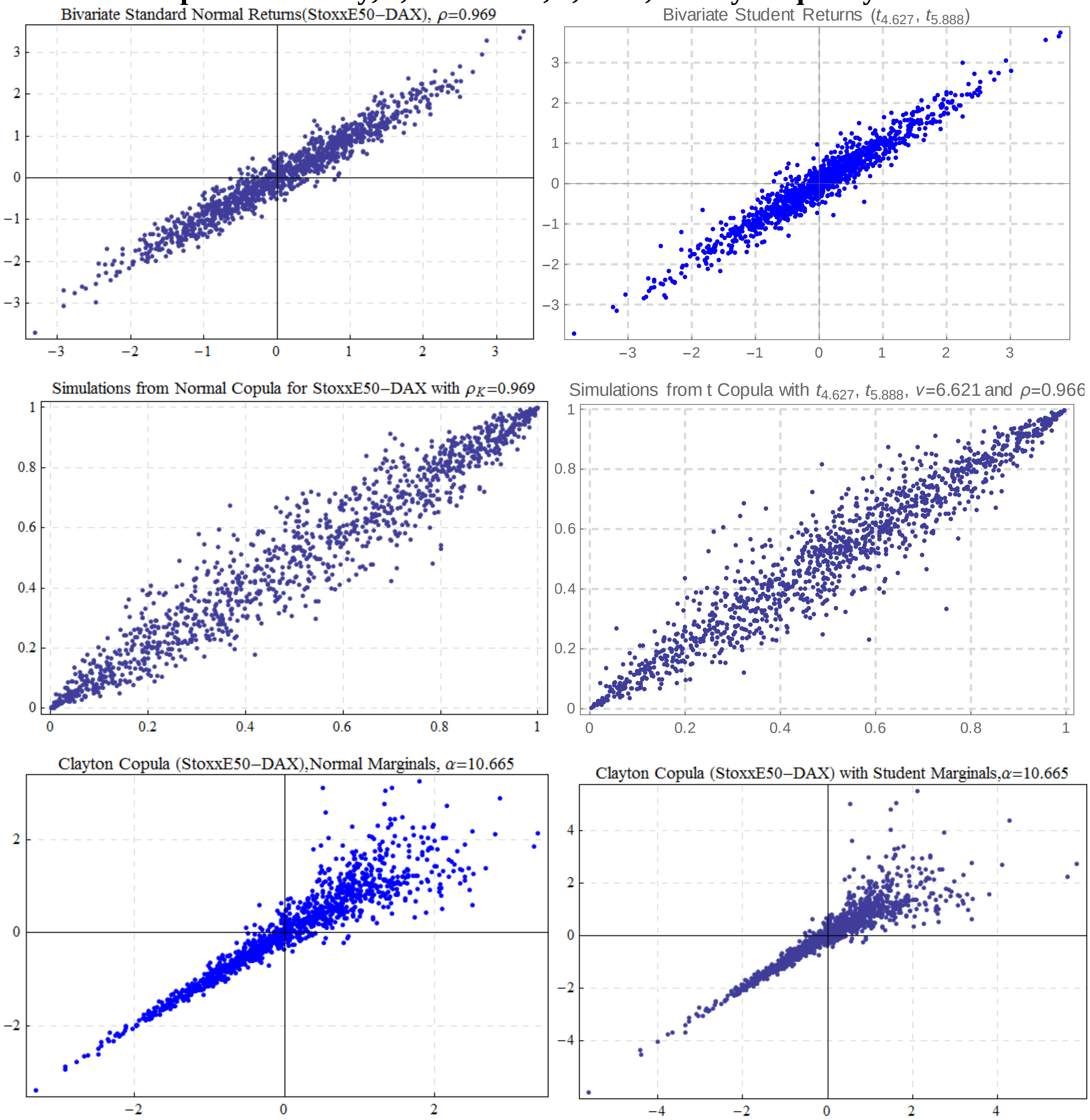
Source: Calculated by authors with Wolfram Mathematica software based on data from http://finance.yahoo.com

Tab. 3: Estimation of the parameters for Student's copulas, for the period January, 7, 1991-June, 2, 2014 , weekly frequency

	MLE estimates of copula parameters	
	Correlation coefficient	**Degree of freedom**
Stoxx50-DAX	0.961	6.6371
Stoxx50-CAC	0.9482	6.6474
CAC-DAX	0.8929	6.6485

Student's t copulas parameters for all pairs of the market indices estimated using MLE of the function (11) we see in table 3. Simulations from normal and Student's t-distribution for pair EuroStoxx 50-DAX we see in figure 3. For comparison we have shown simulations from normal copula with EWMA estimation of the Kendal's correlation coefficient, Student's t copula and Clayton copula with normal and Student's t marginals. We have used EWMA estimation of the parameter θ.

Fig. 3: Bivariate scatter plots for pair StoxxE50-DAX for the period January, 7, 1991-June, 2, 2014, weekly frequency



Conclusion

The problem of the modelling dependencies between financial time series is still open. The paper wants to contribute to the discussion between practitioners and academics about using the copula methodology using time-varying or time invariant model. The time-varying model has been used only for the prediction of correlation coefficients and θ coefficients. The changes of these coefficients reflect the situation on the market. It has been shown that the period when the first crisis began is better detected in evolution of the correlation coefficients. The economic policy adopted by ECB after year 2000 could have caused that the fluctuations of the correlation coefficients were not too extreme. Extreme changes were observed in θ coefficients evolution. It is known that Clayton copula is focused mostly on the dependence in the tails and therefore we assume suitability of Clayton copula using especially during times of crisis or for stress testing during relative stable market situations. Because our data follows Student’s *t* or normal distribution, normal copula focuses on the dependence in the center and exhibits tail independence and the Student's *t* copula captures both central and tail dependence, both copulas can be used for modelling dependencies between selected European market indices. Presence of the normal distribution in the sample data (for example if we take into account data for the last 2 years) may be caused by using weekly frequency. Investors that are typically averse to downside risk must take into account that the problem is not in the model itself but the problem arises from its inappropriate application within the context. They need to know that the risk horizon is given by the frequency of the data. The data with higher frequency do not necessarily exhibit normal distribution, but for the data with lower frequency (weekly data) the classical theory that uses the assumption about normality is still valid.

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