BAYESIAN AND NON-BAYESIAN ANALYSIS FOR THE BURR TYPE XII DISTRIBUTION BASED ON RECORD VALUES AND INTER-RECORD TIMES

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Abstract

In many real life experiments, the only available data is a successive minimum (maximum) that is record-breaking data. In this paper, we obtain the maximum likelihood and Bayesian estimator of the shape parameter for Burr Type XII distribution based on lower record values and inter-record times (the number of trials following the record values) when the other one is known. In the Bayesian case, the estimates are derived under the squared error and the linear-exponential loss functions by using the informative and non informative priors. To be able to generate such a record-breaking data an inverse sampling or random sampling schemes can be used. In this paper, we assume that the record-breaking data are being generated by inverse sampling scheme. By using a Monte Carlo simulation methods: (i) the maximum likelihood and Bayes estimators are compared in terms of the estimated risk, (ii) the estimators are compared with and without the inter-record times are taken into consideration.

Key words: Burr Type XII distribution, record values, inter-record times

JEL Code: C11, C13, C15

Introduction

Let $X_1, X_2, ...$ be a sequence of independent and identical distributed (i.i.d.) continuous random variables. An observation X_j will be called a lower record values if its value is smaller than that of all the previous observations. By definition, X_1 is a lower record value. An analogous definition can be given for upper record values. A record data may be represented by $(\underline{R}, \underline{K}) := (R_1, K_1, R_2, K_2, \ldots, R_m, K_m)$ where R_i is the ith record value, meaning new minimum (or maximum), and K_i is the number trials following the observation of R_i that are needed to obtain a new record value R_{i+1}, which is called inter-record times.

Record values and associated statistics are of great importance in several real life problems involving weather, economic, life-test and sports. In recent years there has been a growing interest in the study of inference problems associated with record values. For example, the Bayesian estimation for the two parameters of some life distributions, including exponential, Weibull, Pareto and Burr Type XII, based on upper record values were considered by (Ahmadi & Doostparast, 2006). Statistical analysis of record values from the Kumaraswamy distribution was considered by (Nadar, Papadopoulos & Kızılaslan, 2013). For more detailed references about the record values see (Arnold, Balakrishnan & Nagaraja, 1998).

Moreover, inference problem with record values and their corresponding inter-record times are recently getting more and more attention. For example, when the underlying distribution is exponential, estimation of the mean parameter was obtained by (Sameniego & Whitaker, 1986) under random sampling scheme and inverse sampling scheme. Non-Bayesian and Bayesian estimates were derived for the two parameters of the exponential distribution based on record values and their corresponding inter-record times under the inverse sampling scheme by (Doostparast, 2009). When the underlying distribution is lognormal, non-Bayesian and Bayesian estimates of the parameters were obtained by (Doostparast, Deepak & Zangoie, 2012).

The two parameter Burr Type XII distribution which was introduced in the literature by (Burr, 1942). Its probability density function (pdf) and cumulative density function (cdf) are given by

$$f(x;\alpha,\beta)=\alpha\beta x^{\alpha-1}(1+x^{\alpha})^{-(\beta+1)},\ x>0, \tag{1}$$

$$F(x;\alpha,\beta)=1-(1+x^{\alpha})^{-\beta} \tag{2}$$

where $\alpha>0$ and $\beta>0$ are the shape parameters and is denoted by $Burr(\alpha,\beta)$. In the literature, a comprehensive studies for the two parameter Burr Type XII distribution were done by several authors. For example, the Bayes estimation of β and the reliability function were obtained by (Papadopoulos, 1978) when α is known. The Bayes estimations of the parameters, reliability and failure rate function based on type II censoring data were considered by (Al-Hussaini & Jaheen 1992, 1995). Under the different loss functions the Bayes estimator of β and reliability function were derived by (Moore & Papadopoulos, 2000). Estimation of the parameters based on generalized order statistics were obtained by (Jaheen, 2005). The empirical Bayes estimation and prediction of β based on record values were discussed by (Wang & Shi, 2010) when α is known. The Bayes estimates of the shape parameters based on the linear exponential loss function were derived by (Nadar & Papadopoulos, 2011).

In this paper, we obtained the estimation of the shape parameter β for the Burr Type XII distribution using lower record values and their corresponding inter-record times under the classical and Bayesian frameworks when the shape parameter α is known. For the sake of comparison we also obtain the estimates based on the lower record values without considering inter-record times. Finally, Monte Carlo simulations are performed to observe the effect of the inter-record times in estimations.

The paper is organized as follows. In Section 1, we derive the maximum likelihood estimation (MLE) of the parameters under the inverse sampling scheme. In Section 2, when the shape parameter α is known, we obtain the Bayesian estimations of β under the symmetric and asymmetric loss functions. In Section 3, a computer simulation study is done to compare the different estimators discussed in early sections and the results are reported. Finally, the paper is completed by a conclusion section.

1 Non-Bayesian Analysis

Under the inverse sampling scheme units are taken sequentially and the sampling is terminated when the m th minimum observation is obtained. In this case, the total number of units sampled is a random number, and K_m is defined to be one for convenience.

In this section, we consider the parameter estimation of Burr Type XII distribution under inverse sampling scheme when the shape parameter α is assumed to be known and unknown.

Let $X_1, X_2, ...$ be independent identical distributed (i.i.d.) random variables, each drawn from a population with cdf $F(.)$ and pdf $f(.)$. Then the likelihood function associated with the sequence $(R_1, K_1, R_2, K_2, \ldots, R_m, K_m)$ is given by (Samaniego & Whitaker, 1986)

$$L(\mathbf{r},\mathbf{k}) = \prod_{i=1}^{m} f(r_i)\{1-F(r_i)\}^{k_i-1} I_{(-\infty, r_{i-1})}(r_i) \tag{3}$$

where $r_0 \equiv \infty$, $k_m \equiv 1$ and $I_A(x)$ is the indicator function of the set A. From the equations (1)-(3), we have

$$L(\alpha,\beta;\mathbf{r},\mathbf{k}) = \alpha^m \beta^m \exp\left\{(\alpha-1)\sum_{i=1}^{m}\ln r_i - \beta\sum_{1=1}^{m} k_i \ln(1+r_i^{\alpha}) - \sum_{i=1}^{m}\ln(1+r_i^{\alpha})\right\}, \tag{4}$$

where $r_1 > \ldots > r_m$. Then, the MLEs of α and β are given by

$$\hat{\beta} = \frac{m}{\sum_{1=1}^{m} K_i \ln(1+R_i^{\hat{\alpha}})} \tag{5}$$

and $\hat{\alpha}$ is the solution of the following non-linear equation

$$\frac{m}{\alpha} + \sum_{i=1}^{m} \frac{\ln r_i}{(1+r_i^{\alpha})} - \frac{m}{\sum_{1=1}^{m} k_i \ln(1+r_i^{\alpha})} \sum_{1=1}^{m} \frac{k_i r_i \ln r_i}{(1+r_i^{\alpha})} = 0 .$$

It can be solved by using the fixed point iteration or Newton-Raphson method.

1.1 MLE estimation of β when α is known

In this case, we assume that α is known and is equal to be α_0 without loss of generality. Then, we have from (4)

$$L(\alpha_0, \beta; \mathbf{r}, \mathbf{k}) = \left(\alpha_0{}^m \prod_{i=1}^{m} \frac{r_i^{\alpha_0 - 1}}{1 + r_i^{\alpha_0}} \right) \beta^m e^{-\beta T(\alpha_0)},\ r_1 > \ldots > r_m \tag{6}$$

where $T(\alpha_0) = \sum_{1=1}^{m} k_i \ln(1+r_i^{\alpha_0})$. It is clear that $T(\alpha_0)$ is a complete sufficient statistic for β and the MLE of β is $\hat{\beta}_M = \frac{m}{T(\alpha_0)}$. The distribution of $\hat{\beta}_M$ can be be obtained by using the moment generating function of $T(\alpha_0)$, which is given as $M(t) = \frac{1}{\left(1 - \frac{t}{\beta}\right)^m}$, $\beta > t$. Therefore, $T(\alpha_0)$ is distributed Gamma with parameters (m, β) with the pdf

$$f(u) = \frac{\alpha^m}{\Gamma(m)} u^{m-1} e^{-\beta u},\ u > 0 .$$

It is easily seen that $E(\hat{\beta}_M) = \frac{m\beta}{m-1}$ and an unbiased estimator of β is given by $\hat{\beta}_U = \frac{m-1}{T(\alpha_0)}$.

Moreover, $\hat{\beta}_U$ is a best unbiased estimator from Lehmann-Scheffé Theorem.

2 Bayesian Analysis

Bayesian approach has a number of advantages over the conventional frequentist approach. Bayes theorem is the only consistent way to modify our beliefs about the parameters given the data that actually occurred. The beliefs about the parameter are called prior distribution. Any prior information about the parameters is considerably useful. We need some prior distributions of the unknown parameters for the Bayesian inference.

In this section, we consider the Bayes estimate of the shape parameter β when the shape parameter α is known. We assume that β has Gamma prior with parameters (a_1+1, b_1) and its pdf is denoted by $\pi(\beta)$. Then, the posterior density function of β is

$$\pi(\beta|\mathbf{r},\mathbf{k}) = \frac{L(\alpha_0,\beta;\mathbf{r},\mathbf{k})\pi(\beta)}{\int_0^\infty L(\alpha_0,\beta;\mathbf{r},\mathbf{k})\pi(\beta)d\beta} = \frac{(b_1+T(\alpha_0))^{m+a_1+1}}{\Gamma(m+a_1+1)}\beta^{m+a_1}e^{-\beta(b_1+T(\alpha_0))}.$$

It means that $\beta|\mathbf{r},\mathbf{k}$ is distributed Gamma with parameters $(m+a_1+1, b_1+T(\alpha_0))$. We know that the Bayes estimate of β under squared error (SE) loss function, $\hat{\beta}_{BS}$, is the mean of the posterior density function of β. Therefore,

$$\hat{\beta}_{BS} = \frac{m+a_1+1}{b_1+T(\alpha_0)} \tag{7}$$

It is well known that the use of symmetric loss functions may be inappropriate in many circumstances, particularly when positive and negative errors have different consequences. A useful alternative to the symmetric loss functions is a convex but asymmetric loss function, called linear-exponential loss function (LINEX), was proposed by (Varian, 1975) and is defined as $L(\theta,\delta) = e^{v(\delta-\theta)} - v(\delta-\theta) - 1,\ v \neq 0$ where δ is an estimator of θ. The sign and magnitude of v represents the direction and degree of asymmetry, respectively. If $v > 0$, the overestimation is more serious than underestimation, and vice versa. For v close to zero, the LINEX loss is approximately SE loss and almost symmetric. It is easily seen that the value of $\delta(X)$ that minimizes $E_{\theta|X}\left[L(\theta,\delta(X))\right]$ is $\hat{\delta}_{BL} = -\frac{1}{v}\ln\left(E_{\theta|X}(e^{-v\theta})\right)$, provided $E_{\theta|X}(e^{-v\theta})$ exists and is finite. Here $E_{\theta|X}(.)$ denotes the expected value with respect to the posterior density function θ given X.

The Bayes estimator of β under the LINEX loss function, $\hat{\beta}_{BL}$, is obtained as

$$\hat{\beta}_{BL} = -\frac{1}{v}\ln E_{\beta|(\mathbf{r},\mathbf{k})}(e^{-v\beta}) = \frac{m+a_1+1}{v}\ln\left(1+\frac{v}{b_1+T(\alpha_0)}\right). \tag{8}$$

If we use the Jeffrey's non-informative prior, $\pi(\beta) \propto 1/\beta$, then we have $\beta | \mathbf{r}, \mathbf{k}$ is distributed Gamma with parameters $(m, T(\alpha_0))$. Therefore, the Bayes estimates of β under the SE and LINEX loss functions are obtained as

$$\hat{\beta}_{BS,0} = \frac{m}{T(\alpha_0)}, \quad \hat{\beta}_{BL,0} = \frac{m}{v} \ln\left(1 + \frac{v}{T(\alpha_0)}\right). \tag{9}$$

3 Simulation Study

In order to compare the different estimators, Monte Carlo simulations are performed by using different sample sizes and different priors. All the programs are written in Matlab 2010a. All the results are based on 1000 replications. The estimated risk (ER) of θ, when θ is estimated

Tab. 1: Estimations of β and ERs for $\alpha = 3$ when the inter-record times are considered.

(a_1, b_1)	m	β	$\hat{\beta}_M$	$\hat{\beta}_U$	$\hat{\beta}_{BS}$	$\hat{\beta}_{BL}$				
						v	-2	-1	1	2
(4,6)	3	0.8284	1.2490	0.8327	0.8330		0.9435	0.8834	0.7900	0.7526
			1.4670	0.5575	0.0992		0.2407	0.0543	0.0458	0.1708
	5		1.0446	0.8357	0.8368		0.9249	0.8777	0.8010	0.7691
			0.4246	0.2374	0.0774		0.1772	0.0416	0.0361	0.1349
	8		0.9452	0.8270	0.8301		0.8960	0.8612	0.8020	0.7765
			0.1995	0.1411	0.0626		0.1405	0.0332	0.0297	0.1134
	10		0.9220	0.8298	0.8299		0.8869	0.8570	0.8052	0.7825
			0.1324	0.0974	0.0495		0.1038	0.0254	0.0242	0.0943
(5,5)	3	1.2086	1.8828	1.2552	1.2034		1.4147	1.2970	1.1264	1.0615
			8.0253	3.3170	0.1868		0.5007	0.1072	0.0832	0.3020
	5		1.5495	1.2396	1.2011		1.3714	1.2781	1.1359	1.0796
			1.3788	0.7850	0.1510		0.3393	0.0811	0.0696	0.2574
	8		1.4030	1.2277	1.2162		1.3498	1.2779	1.1623	1.1146
			0.3510	0.2367	0.1191		0.2704	0.0636	0.0557	0.2096
	10		1.3511	1.2160	1.2070		1.3210	1.2602	1.1599	1.1176
			0.3155	0.2366	0.1063		0.2354	0.0562	0.0500	0.1872

Source: own computations

by $\hat{\theta}$, is given by $ER(\theta)=\frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i-\theta_i)^2$ under the SE loss function. Moreover, the estimated risk of θ under the LINEX loss function is given by

$$ER(\theta)=\frac{1}{N}\sum_{i=1}^{N}\left(e^{v(\hat{\theta}_i-\theta_i)}-v(\hat{\theta}_i-\theta_i)-1\right).$$

In the Table 1, we consider the case where $\alpha=3$ and β has Gamma prior with parameter $(a_1,b_1)=(4,6)$ and $(a_1,b_1)=(5,5)$. When the estimates obtained without taking inter-record times into consideration, the results are given in Table 2 and is denoted by β^*. The ML and Bayesian estimates for SE and LINEX loss functions are listed in both Tables 1 and 2.

Tab. 2: Estimations of β and ERs for $\alpha=3$ when the inter-record times are not considered.

(a_1,b_1)	m	β	$\hat{\beta}^*_M$	$\hat{\beta}^*_{BS}$	$\hat{\beta}^*_{BL}$				
					v	-2	-1	1	2
(4,6)	3	0.8284	1.5451	0.5557		0.4911	0.4959	0.8386	0.8434
			5.1532	0.1937		1.0969	0.1640	0.0542	0.2225
	5		1.4989	0.5577		0.4926	0.4975	0.8416	0.8466
			4.6678	0.1926		1.0937	0.1634	0.0539	0.2196
	8		1.5275	0.5581		0.4921	0.4976	0.8426	0.8481
			4.4683	0.1899		1.0839	0.1616	0.0526	0.2156
	10		1.4775	0.5559		0.4913	0.4961	0.8389	0.8436
			3.6899	0.1937		1.0968	0.1641	0.0543	0.2233
(5,5)	3	1.2086	2.5249	0.8372		0.6375	0.7117	1.2986	1.3729
			31.1836	0.3644		4.3029	0.3936	0.1155	0.6740
	5		2.4101	0.8303		0.6351	0.7068	1.2869	1.3586
			26.9995	0.3683		4.3133	0.3959	0.1132	0.6480
	8		2.6684	0.8375		0.6360	0.7116	1.2994	1.3750
			25 .2401	0.3594		4.3083	0.3895	0.1125	0.6623
	10		2.6460	0.8414		0.6383	0.7146	1.3058	1.3821
			16.4012	0.3607		4.2984	0.3901	0.1164	0.6851

Source: own computations

Note: In the Tables, the first and second rows represent the average estimates and the estimated risks.

In Tables 1 and 2, it is observed that as the sample size increases the estimated risk of the estimates generally decrease. The ERs of the MLEs are greatest among all estimators. Moreover, the ERs of the Bayes estimators under the SE loss function are smaller than the MLEs, as expected. Furthermore, it is observed that the ERs for estimates of β are smaller than that of β^*. It is quite natural to see such a result when more information is available. The simulation illustrates that considering inter-record times is increasing the accuracy and the precision of the estimates.

Conclusion

In this study, we compared the different estimators of the shape parameter β for the Burr Type XII distribution when the shape parameter α is known. It is observed that the Bayesian estimators have a smaller estimated risk and this result does not change for the different values of the prior parameters by using Monte Carlo simulation. Moreover, the simulation illustrates that taking the inter-record times into consideration increases the accuracy and the precision of the estimators.

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