# GAUSSIAN AND MEAN CURVATURE STUDIED WITH THE HELP OF CARTAN MOVING FRAME AND WEINGARTEN MAP

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**Abstract**

The goal of this paper is the deduction of two famous formulas concerning classical differential geometry and deduce them for general form of Monge surfaces. One of them is the Gaussian curvature formula, which can be constructed only with the use of Maurer-Cartan equations. These equations are based on Cartan's idea of moving frame, which is used in many parts of mathematics, especially in differential geometry. The other part of the text is devoted to Mean curvature formula deduced with the use of Cartan moving frame and Weingarten map. Thanks to these methods, not only Mean, but also Gaussian curvature formula can be deduced. Studying of Gaussian curvature in general form on Monge surfaces only with the use of Maurer-Cartan equations is not a simple problem but essential in differential geometry. Deduction of Mean curvature with the use of Weingarten map of given general Monge surface is essential in geometry, too. Gaussian and Mean curvature formulas of general Monge surface are common results however, methods of their deduction are especially essential.

**Key words:** Gaussian and Mean curvature, moving frame, Maurer-Cartan equations, differential forms, differentiable homeomorphism, Weingarten map.

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## Introduction

The article consists of two chapters. The first is devoted to the construction the formula of Gaussian curvature. The main reason why the first chapter is interesting follows from the fact that this work is based only on Maurer-Cartan equations formed with the use of orthonormal moving frame. The method of Maurer-Cartan equations is very time consuming but very interesting.

In the second chapter Mean curvature is deduced with the use of Weingaeten map and also Cartan moving frame. Both chapters are based on using the regular mappings, regular two dimensional surfaces in $R^3$, differentiable homeomorphisms etc.

Prominent authors who studied these problems and whose work influenced this article were for instance these great scientists: Kobayashi S. & Nomizu K. in their famous work "Foundations of Differential Geometry" (Kobayashi, Nomizu, 1963; Kobayashi, Nomizu, 1969) and Sternberg S. in his excellent book "Lectures on Differential Geometry" (Sternberg, 1964). Presented methods may be also useful in econophysics. The contemporary author who is interested in econophysics and uses some of differential geometry methods is T. Zeithamer (Zeithamer, 2012a; Zeithamer, 2012b).

Let $U \subset R^2$ be an open neighbourhood of a point $(u, v) \in U$ and $x: U \to R^3$ a regular map. A subset $M \subset R^3$ is called a regular two dimensional surface in $R^3$ if for each $p \in M$ there exist an open neighbourhood $V$ of $p \in R^3$ and a map $x: U \to V \cap M$ of an open set $U \subset R^2$ onto $V \cap M$ such that $x$ is a differentiable homeomorphism and the differential $dx_q: T_q(U) \to T_{x(q)}(M)$ is injective for all $q \in U, x(q) = p$. It is possible to choose in $x(U)$ an orthonormal moving frame $\{E_1, E_2, E_3\}$ in such a way that $E_1, E_2$ are tangent to $x(U)$ and $E_3$ is a non-vanishing normal to $x(U)$.

## Chapter I. Gaussian curvature formula for general Monge surfaces

Moving frame for the general Monge surface $x(u,v)= (u,v,h(u,v))$ where $x: U \to R^3$, $U$ is an open set in $R^2$ and $h: U \to R$ is a differentiable function. Mooving frame of the map $x$ has the form

$$x_u = (1, 0, h_u), \quad x_v = (0, 1, h_v) \quad n = (-h_u, -h_v, 1).$$

Using the Gramm-Schmidt orthonormalization process, we obtain orthonormal moving frame of the form

$$E_1 = \left(\frac{1}{\sqrt{1+h_u^2}}, 0, \frac{h_u}{\sqrt{1+\ h_u^2}}\right),$$

$$E_2 = \left(\frac{-h_u h_v}{\sqrt{1+h_u^2}\sqrt{1+h_u^2+\ h_v^2}}, \frac{1+\ h_u^2}{\sqrt{1+h_u^2}\sqrt{1+h_u^2+\ h_v^2}}, \frac{h_v}{\sqrt{1+h_u^2}\sqrt{1+h_u^2+\ h_v^2}}\right),$$

$$E_3 = \left(\frac{-h_u}{\sqrt{1+h_u^2+\ h_v^2}}, \frac{-h_v}{\sqrt{1+h_u^2+\ h_v^2}}, \frac{1}{\sqrt{1+h_u^2+\ h_v^2}}\right).$$

It can be easily verified that

$$\partial_u E_1 = \left( \frac{-h_u h_{uu}}{(1 + h_u^2)^{\frac{3}{2}}}, 0, \frac{h_{uu}}{(1 + h_u^2)^{\frac{3}{2}}} \right), \quad \partial_v E_1 = \left( \frac{-h_u h_{uv}}{(1 + h_u^2)^{\frac{3}{2}}}, 0, \frac{h_{uv}}{(1 + h_u^2)^{\frac{3}{2}}} \right).$$

See (Kaňka, 2012; Kaňka, Kaňková, 2011).

So we have

$$dE_1 = \left( \frac{-h_u h_{uu}}{(1 + h_u^2)^{\frac{3}{2}}}, 0, \frac{h_{uu}}{(1 + h_u^2)^{\frac{3}{2}}} \right) du + \left( \frac{-h_u h_{uv}}{(1 + h_u^2)^{\frac{3}{2}}}, 0, \frac{h_{uv}}{(1 + h_u^2)^{\frac{3}{2}}} \right) dv,$$

$$\omega_{12} = dE_1 \cdot E_2 = \left( \frac{h_u^2 h_v h_{uu} + h_{uu} h_v}{(1 + h_u^2)^2 \cdot \sqrt{1 + h_u^2 + h_v^2}} \right) du + \left( \frac{h_u^2 h_v h_{uv} + h_{uv} h_v}{(1 + h_u^2)^2 \cdot \sqrt{1 + h_u^2 + h_v^2}} \right) dv.$$

Simplifying the previous result, we have

$$\omega_{12} = \frac{h_{uu} h_v}{(1 + h_u^2)\sqrt{1 + h_u^2 + h_v^2}} du + \frac{h_v h_{uv}}{(1 + h_u^2)\sqrt{1 + h_u^2 + h_v^2}} dv.$$

It can easily be proved that the equation $\omega_{12} = -\omega_{21}$ is satisfied. See (Bureš, Kaňka, 1994).

The result of $dE_1$ gives

$$\omega_{13} = dE_1 \cdot E_3 = \frac{h_u^2 h_{uu} + h_{uu}}{(1 + h_u^2)^{\frac{3}{2}}\sqrt{1 + h_u^2 + h_v^2}} du + \frac{h_u^2 h_{uv} + h_{uv}}{(1 + h_u^2)^{\frac{3}{2}}\sqrt{1 + h_u^2 + h_v^2}} dv,$$

from which follows

$$\omega_{13} = \frac{h_{uu}}{\sqrt{1 + h_u^2}\sqrt{1 + h_u^2 + h_v^2}} du + \frac{h_{uv}}{\sqrt{1 + h_u^2}\sqrt{1 + h_u^2 + h_v^2}} dv.$$

Short calculation gives

$$\partial_u E_3 \cdot E_1 = -\frac{h_{uu}}{\sqrt{1 + h_u^2}\sqrt{1 + h_u^2 + h_v^2}}, \quad \partial_v E_3 \cdot E_1 = -\frac{h_{uv}}{\sqrt{1 + h_u^2}\sqrt{1 + h_u^2 + h_v^2}}.$$

The form $\omega_{31}$ is

$$\omega_{31} = dE_3 \cdot E_1 = -\frac{h_{uu}}{\sqrt{1 + h_u^2}\sqrt{1 + h_u^2 + h_v^2}} du - \frac{h_{uv}}{\sqrt{1 + h_u^2}\sqrt{1 + h_u^2 + h_v^2}} dv.$$

See (Bochner, 1951; Kostant, 1956a; Kostant, 1956b).

Formulas for $\partial_u E_3$ and $\partial_v E_3$ are

$$\partial_u E_3 = \left(\frac{-h_{uu}(1+\ h_v^2)+\ h_u h_v h_{uv}}{(1+\ h_u^2+\ h_v^2)^{\frac{3}{2}}}, \frac{-h_{uv}(1+\ h_u^2)+\ h_u h_v h_{uu}}{(1+\ h_u^2+\ h_v^2)^{\frac{3}{2}}}, \frac{-h_u h_{uu}-\ h_v h_{uv}}{(1+\ h_u^2+\ h_v^2)^{\frac{3}{2}}}\right),$$

$$\partial_v E_3 = \left(\frac{-h_{uv}(1+\ h_v^2)+\ h_u h_v h_{vv}}{(1+\ h_u^2+\ h_v^2)^{\frac{3}{2}}}, \frac{-h_{vv}(1+\ h_u^2)+\ h_u h_v h_{uv}}{(1+\ h_u^2+\ h_v^2)^{\frac{3}{2}}}, \frac{-h_u h_{uv}-\ h_v h_{vv}}{(1+\ h_u^2+\ h_v^2)^{\frac{3}{2}}}\right).$$

By analogy,

$$\partial_u E_3\ \cdot\ E_2 = \frac{h_u h_v h_{uu}-\ h_{uv}(1+h_u^2)}{\sqrt{1+h_u^2}(1+\ h_u^2+\ h_v^2)}, \quad \partial_v E_3\ \cdot\ E_2 = \frac{h_u h_v h_{uv}-\ h_{vv}(1+h_u^2)}{\sqrt{1+h_u^2}(1+\ h_u^2+\ h_v^2)}.$$

So we have

$$\omega_{32} = dE_3 \cdot E_2 = \frac{h_u h_v h_{uu} - h_{uv}(1+h_u^2)}{\sqrt{1+h_u^2}(1+h_u^2+h_v^2)}\, du + \frac{h_u h_v h_{uv} - h_{vv}(1+h_u^2)}{\sqrt{1+h_u^2}(1+h_u^2+h_v^2)}\, dv.$$

Forms $\theta_1 = E_1 dx = E_1 x_u\, du + E_1 x_v\, dv$ and $\theta_2 = E_2\, dx = E_2\, x_u\, du + E_2\, x_v\, dv$ are

$$\theta_1 = \left(\frac{1}{\sqrt{1+h_u^2}}, 0, \frac{h_u}{\sqrt{1+h_u^2}}\right)(1,0,h_u)du + \left(\frac{1}{\sqrt{1+h_u^2}}, 0, \frac{h_u}{\sqrt{1+h_u^2}}\right)(0,1,h_v)dv =$$
$$= \sqrt{1+h_u^2}du + \frac{h_u h_v}{\sqrt{1+h_u^2}}dv,$$

and, by a similar calculation,

$$\theta_2 = \frac{\sqrt{1+h_u^2+h_v^2}}{\sqrt{1+h_u^2}}dv.$$

The exterior product $\theta_1 \wedge \theta_2$ can be written in the form

$$\theta_1 \wedge \theta_2 = \sqrt{1+h_u^2+h_v^2}\, du \wedge dv. \tag{1}$$

First results:

$$\omega_{12} = \frac{h_{uu} h_v du + h_v h_{uv} dv}{(1+h_u^2)\sqrt{1+h_u^2+h_v^2}}, \quad \omega_{21} = \frac{-h_{uu} h_v du - h_v h_{uv} dv}{(1+h_u^2)\sqrt{1+h_u^2+h_v^2}},$$

$$\omega_{13} = \frac{h_{uu}du + h_{uv}dv}{\sqrt{1+h_u^2}\sqrt{1+h_u^2+h_v^2}}, \quad \omega_{31} = \frac{-h_{uu}du - h_{uv}dv}{\sqrt{1+h_u^2}\sqrt{1+h_u^2+h_v^2}},$$

$$\omega_{32} = \frac{[h_u h_v h_{uu} - h_{uv}(1+h_u^2)]du + [h_u h_v h_{uv} - h_{vv}(1+h_u^2)]dv}{\sqrt{1+h_u^2}(1+h_u^2+h_v^2)},$$

$$\omega_{31} \wedge \omega_{32} = \frac{h_{uu}h_{vv} - h_{uv}^2}{(1+h_u^2+h_v^2)^{3/2}} du \wedge dv. \qquad (2)$$

Expressing $du \wedge dv$ from (1) and substituting into (2) , we obtain

$$\omega_{31} \wedge \omega_{32} = \frac{h_{uu}h_{vv} - h_{uv}^2}{(1+h_u^2+h_v^2)^2} \theta_1 \wedge \theta_2.$$

The formula $\omega_{31} \wedge \omega_{32} = K \cdot \theta_1 \wedge \theta_2$ gives the expression for the Gaussian curvature

$$K = \frac{h_{uu}h_{vv} - h_{uv}^2}{(1+h_u^2+h_v^2)^2}.$$

## Chapter II. Gaussian and Mean curvature formula for general Monge surfaces

he equation $E_3 \cdot E_3 = 1$ gives $\partial_u(E_3 \cdot E_3) = 2\partial_u E_3 \cdot E_3 = 0$, which means that $\partial_u E_3 \in T_p(M)$. Analogously $\partial_v E_3 \in T_p(M)$. So we have

$$\partial_u E_3 = c_{11}x_u + c_{12}x_v, \quad \partial_v E_3 = c_{21}x_u + c_{22}x_v.$$

Multiplying the first equation by $x_u$ and $x_v$ , we obtain

$$\partial_u E_3 \cdot x_u = c_{11}x_u \cdot x_u + c_{12}x_v \cdot x_u, \qquad (3)$$

$$\partial_u E_3 \cdot x_v = c_{11}x_u \cdot x_v + c_{12}x_v \cdot x_v. \qquad (4)$$

See (Rauch, 1951).

The formula

$$\partial_u E_3 = \left( \frac{h_u h_v h_{uv} - h_{uu}(1 + h_v^2)}{(1 + h_u^2 + h_v^2)^{3/2}}, \quad \frac{h_u h_v h_{uu} - h_{uv}(1 + h_u^2)}{(1 + h_u^2 + h_v^2)^{3/2}}, \quad -\frac{h_u h_{uu} + h_v h_{uv}}{(1 + h_u^2 + h_v^2)^{3/2}} \right)$$

gives, after a short calculation,

$$\partial_u E_3 \cdot x_u = -\frac{h_{uu}}{\sqrt{1 + h_u^2 + h_v^2}}, \quad \partial_u E_3 \cdot x_v = -\frac{h_{uv}}{\sqrt{1 + h_u^2 + h_v^2}}.$$

Substituting into (3) and (4), we arrive at the system of equations

$$-\frac{h_{uu}}{\sqrt{1 + h_u^2 + h_v^2}} = c_{11}(1 + h_u^2) + c_{12} h_u h_v,$$
$$-\frac{h_{uv}}{\sqrt{1 + h_u^2 + h_v^2}} = c_{11} h_u h_v + c_{12}(1 + h_v^2),$$

which can be solved. Using the Cramer rule, we obtain

$$D = \begin{vmatrix} 1 + h_u^2 & h_u h_v \\ h_u h_v & 1 + h_v^2 \end{vmatrix} = 1 + h_u^2 + h_v^2,$$

$$D_1 = \begin{vmatrix} -\frac{h_{uu}}{\sqrt{1 + h_u^2 + h_v^2}} & h_u h_v \\ -\frac{h_{uv}}{\sqrt{1 + h_u^2 + h_v^2}} & 1 + h_v^2 \end{vmatrix} = \frac{-h_{uu}(1 + h_v^2) + h_u h_v h_{uv}}{\sqrt{1 + h_u^2 + h_v^2}},$$

$$D_2 = \begin{vmatrix} 1 + h_u^2 & -\frac{h_{uu}}{\sqrt{1 + h_u^2 + h_v^2}} \\ h_u h_v & -\frac{h_{uv}}{\sqrt{1 + h_u^2 + h_v^2}} \end{vmatrix} = \frac{-h_{uv}(1 + h_u^2) + h_u h_v h_{uu}}{\sqrt{1 + h_u^2 + h_v^2}},$$

$$c_{11} = \frac{D_1}{D} = \frac{h_u h_v h_{uv} - h_{uu}(1 + h_v^2)}{(1 + h_u^2 + h_v^2)^{3/2}}, \quad c_{12} = \frac{D_2}{D} = \frac{h_u h_v h_{uu} - h_{uv}(1 + h_u^2)}{(1 + h_u^2 + h_v^2)^{3/2}}.$$

By analogy, multiplying the equation $\partial_v E_3 = c_{21} x_u + c_{22} x_v$ by $x_u$ and $x_v$ we obtain

$$\partial_v E_3 \cdot x_u = c_{21} x_u \cdot x_u + c_{22} x_v \cdot x_u, \tag{5}$$

$$\partial_v E_3 \cdot x_v = c_{21} x_u \cdot x_v + c_{22} x_v \cdot x_v. \tag{6}$$

The relation

$$\partial_v E_3 = \left( \frac{h_u h_v h_{vv} - h_{uv}(1+h_v^2)}{(1+h_u^2+h_v^2)^{3/2}}, \quad \frac{h_u h_v h_{uv} - h_{vv}(1+h_u^2)}{(1+h_u^2+h_v^2)^{3/2}}, \quad -\frac{h_u h_{uv} + h_v h_{vv}}{(1+h_u^2+h_v^2)^{3/2}} \right)$$

then yields

$$\partial_v E_3 \cdot x_u = -\frac{h_{uv}}{\sqrt{1+h_u^2+h_v^2}}, \quad \partial_v E_3 \cdot x_v = -\frac{h_{vv}}{\sqrt{1+h_u^2+h_v^2}} \, .$$

Substituting into (5) and (6), our task now is to solve the system of equations

$$-\frac{h_{uv}}{\sqrt{1+h_u^2+h_v^2}} = c_{21}(1+h_u^2) + c_{22} h_u h_v,$$
$$-\frac{h_{vv}}{\sqrt{1+h_u^2+h_v^2}} = c_{21} h_u h_v + c_{22}(1+h_v^2).$$

Using Cramer's rule, we obtain

$$D = \begin{vmatrix} 1+h_u^2 & h_u h_v \\ h_u h_v & 1+h_v^2 \end{vmatrix} = 1 + h_u^2 + h_v^2,$$

$$D_1 = \begin{vmatrix} -\frac{h_{uv}}{\sqrt{1+h_u^2+h_v^2}} & h_u h_v \\ -\frac{h_{vv}}{\sqrt{1+h_u^2+h_v^2}} & 1+h_v^2 \end{vmatrix} = \frac{-h_{uv}(1+h_v^2) + h_u h_v h_{vv}}{\sqrt{1+h_u^2+h_v^2}},$$

$$D_2 = \begin{vmatrix} 1+h_u^2 & -\frac{h_{uv}}{\sqrt{1+h_u^2+h_v^2}} \\ h_u h_v & -\frac{h_{vv}}{\sqrt{1+h_u^2+h_v^2}} \end{vmatrix} = \frac{-h_{vv}(1+h_u^2) + h_u h_v h_{uv}}{\sqrt{1+h_u^2+h_v^2}}.$$

So we have

$$c_{21} = \frac{h_u h_v h_{vv} - h_{uv}(1+h_v^2)}{(1+h_u^2+h_v^2)^{3/2}}, \quad c_{22} = \frac{h_u h_v h_{uv} - h_{vv}(1+h_u^2)}{(1+h_u^2+h_v^2)^{3/2}}.$$

The matrix *A* consisting of coefficients $c_{11}$, $c_{12}$, $c_{21}$, $c_{22}$ equals

$$A = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} \frac{h_u h_v h_{uv} - h_{uu}(1+h_v^2)}{(1+h_u^2+h_v^2)^{3/2}} & \frac{h_u h_v h_{uu} - h_{uv}(1+h_u^2)}{(1+h_u^2+h_v^2)^{3/2}} \\ \frac{h_u h_v h_{vv} - h_{uv}(1+h_v^2)}{(1+h_u^2+h_v^2)^{3/2}} & \frac{h_u h_v h_{uv} - h_{vv}(1+h_u^2)}{(1+h_u^2+h_v^2)^{3/2}} \end{pmatrix}.$$

The Gaussian curvature equals to the det *A*

$$K = \det A = \frac{h_{uu}h_{vv} - h_{uv}^2}{(1 + h_u^2 + h_v^2)^2},$$

as was given previously. The Mean curvature equals to the $-\frac{1}{2}\operatorname{tr} A$ .

$$H = -\frac{1}{2}\operatorname{tr} A = \frac{h_{uu}(1 + h_v^2) - 2h_u h_v h_{uv} + h_{vv}(1 + h_u^2)}{2(1 + h_u^2 + h_v^2)^{3/2}}.$$

## Conclusion

The first main result of the text is the fact that the deduction of Gaussian curvature *K* of general Monge surface

$$K = \frac{h_{uu}h_{vv} - h_{uv}^2}{(1 + h_u^2 + h_v^2)^2}$$

can be done only with the use of Maurer-Cartan equations.

The second main result is that the Mean curvature *H*

$$H = -\frac{1}{2}\operatorname{tr} A = \frac{h_{uu}(1 + h_v^2) - 2h_u h_v h_{uv} + h_{vv}(1 + h_u^2)}{2(1 + h_u^2 + h_v^2)^{3/2}}$$

can be deduced with the use of Cartan moving frame and Weingarten map.

## References


Bochner, S. (1951). A new viewpoint in differential geometry. *Canadian Journal of mathematics – Journal Canadian de Mathematiques*, *3*(4), 460–470. Retrieved from http://cms.math.ca/cjm/v3/cjm1951v03.0460-0470.pdf

Bureš, J. & Kaňka, M. (1994). Some Conditions for a Surface in to be a Part of the Sphere. *Mathematica Bohemica*, *119*(4), 367–371. Retrieved from http://dml.cz/bitstream/handle/10338.dmlcz/126121/MathBohem_119-1994-4_4.pdf

Kaňka, M. (2012). Application of Maurer-Cartan Equations. *Mundus Symbolicus, 20*(20), 27–37.

Kaňka, M. & Kaňková, E. (2011, 09). Ing. Tomáš Pavelka (Chair). *Geometry one special type of surfaces in R-3*. International days of statistics and economics, Praha.

Kobayashi, S. & Nomizu, K. (1963). *Foundations of differential geometry*. (1st ed., Vol.1). New York-London-Sydney: Interscience Publishers - John Wiley.

Kobayashi, S. & Nomizu, K. (1969). *Foundations of differential geometry*. (1st ed., Vol.2). New York-London-Sydney: Interscience Publishers - John Wiley.

Kostant, B. (1956a). On differential geometry and homogeneous space. 1. *Proceedings of the national Academy of sciences of the United States of America, 42*(5), 258–261. Retrieved from http://www.pnas.org/content/42/5/258

Kostant, B. (1956b). On differential geometry and homogeneous space. 2. *Proceedings of the national Academy of sciences of the United States of America, 42*(6), 354–357. Retrieved from http://www.pnas.org/content/42/6/354

Rauch, H. (1951). A contribution to differential geometry in the large. *Annals of Mathematics, 54*(1), 38–55. Retrieved from http://www.jstor.org.ezproxy.vse.cz/stable/1969309?seq=1

Sternberg, S. (1964). *Lectures on differential geometry*. (2nd ed.). Englewood Cliffs, New Jersy: Prentice-Hall, INC.

Zeithamer, T. (2012a, June). In Aitekin Isman (Chair). *Analytical theory of monotone commodity state development with inflection*. International conference on new horizonts education 2012, Czech Republic - Prague. doi: 10.1016/j.sbspro.2012.09.523

Zeithamer, T. (2012b, June). In Aitekin Isman (Chair). *Economic Phenomena from the Viewpoint of the Mechanics of Matherials*. International conference on new horizonts education 2012, Czech Republic - Prague. doi: 10.1016/j.sbspro.2012.09.53

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