SKEWED-EWMA FORECAST OF THE VAR USING COPULA APPROACH

Mária Bohdalová – Michal Greguš

Abstract

In this paper we solve the problem of estimating the risk of portfolios with complex dependencies. Recently, Lu, Huang and Gerlach (2010) proposed a skewed–*EWMA* estimator to calculate Value–at–Risk (*VaR*) for individual financial assets that take into account their volatilities. Skewed–*EWMA* estimator is derived from the asymmetric Laplace distribution and considers both skewness and heavy tails of the return distribution. This method is adaptive to the time-varying nature in practice by adjusting shape parameter in the distribution. In this paper, we extend the skewed–*EWMA* procedure introduced by Lu and Li (2011) to estimate the risk of complex portfolios with dependencies modeled via Student t copula and Clayton copula. We develop Monte Carlo simulation procedure, which combines copula techniques with skewed–*EWMA* forecasting of the volatility of the risk factors and asymmetric Laplace distribution. A portfolio composed from European stock indices illustrates our proposed method.

Key words: elliptical copula, Monte Carlo simulation, skewed *EWMA VaR*, portfolio risk

JEL Code: C15, G11, G17, G32

Introduction

Value–at–Risk (*VaR*) is probably the most widely used risk measure in financial institutions. It has also made its way into the Basel II capital–adequacy framework (McNeil et al., 2005), that provides a way of quantifying and managing the risk of a portfolio. Let us have some portfolio of risky assets and a fixed time horizon Δ and let us denote the distribution function of the corresponding loss distribution by $F_L(l) = P(L \leq l)$. *VaR* is a measure of the risk based on F_L which measures the severity of the risk of holding our portfolio over the time period Δ. In most models of interest the support of F_L is unbounded so that the maximum loss is simply infinity. Value–at–Risk is a "maximum loss which is not exceeded with a given high probability", the so–called confidence level $\alpha \in (0, 1)$. The VaR_α of our portfolio at the confidence level α is given by the smallest number l such, that the probability that the loss L exceeds l is not larger than $(1 - \alpha)$. Formally,

$$VaR_\alpha = inf\{l \in R : P(L > l) \leq 1 - \alpha\} = \inf\{l \in R : F_L(l) \geq \alpha\}. \quad (1)$$

In probabilistic terms, VaR_α is an α–quantile of the loss distribution (McNeil et al., 2005). Typical values for α are $\alpha = 0.95$ or $\alpha = 0.99$; in market risk management the time horizon Δ is usually one or ten days, in credit risk management and operational risk management Δ is usually one year.

There are different methods to calculate *VaR* like variance–covariance method, historical simulation and Monte Carlo simulation method. The Monte Carlo method is a rather general name for any approach to risk measurement that involves the simulation of an explicit parametric model for risk–factor changes. As such, the method can be either conditional or unconditional depending on the fact whether the model adopted is a dynamic time series model for risk–factor changes or a static distributional model (Dowd, 2002), (Dempster, Howard, 2002), (McNeil et al., 2005), (Alexander, 2008), (Jorion, 2009).

The first step of the method is the choice of the model and the calibration of this model to the historical risk–factor change data. Obviously it should be a model from which we can readily simulate m independent realizations of risk–factor changes for the next time period. If we apply the loss operator to these simulated vectors then we obtain simulated realizations from the loss distribution F_L. These simulated loss data are used to estimate risk measures. We can choose the number of replications m ourselves, within the obvious constraints of computation time. Generally speaking, with larger m we obtain greater accuracy *VaR* (McNeil et al., 2005).

Due to the facts present in financial time series, a key problem in financial research, and particularly in the field of risk management, is the choice of the models for risk factors so as to avoid systematic biases in the measurement of risk. Risk factor models can be fitted using different distributional specifications, including non–normal distributions. This multivariate simulation process captures and maintains the dependence structure of the risk factors modeled separately. To accomplish this, the simulation engine uses a framework based on the statistical concept of a copula (Embrechts et al., 2001), (McNeil et al., 2005). A copula is a function that combines marginal distributions of the variables (risk factors) into a specific multivariate distribution in which all of its one dimensional marginal are cumulative distribution functions (*CDFs*) of the risk factors (Bohdalová, 2007).

Nowadays the distribution of the financial risk–factor return series is often skewed and heavy–tailed with departure from normality. Lu, Huang and Gerlach were motivated by an asymmetric Laplace distribution (*ALD*) to take into account both skewness and heavy tails in financial return distributions (Lu, Huang, Gerlach, 2010). As a result, they created the skewed–*EWMA* procedure to forecast the changing volatility. In this paper we introduce Monte Carlo simulation method of *VaR* for marginal with asymmetric Laplace distribution $AL(\mu,\sigma,p)$ and we compare it with Monte Carlo simulation method of *VaR* with Student *t* marginal. Finally, we use Student *t* (elliptical) copula and Clayton copula to compute *VaR* of the portfolio.

The outline of the paper is as follows. The following section introduces methodology of the asymmetric Laplace distribution, skewed–*EWMA* forecast of volatility for marginal and algorithm for estimating VaR_α using copula methodology. Section two presents the empirical results and discussion. Section three concludes our findings.

1 Methodology

Due to the dynamics of financial markets it is more problematic to estimate the risk of portfolios. The empirical distributions of the risk factors appear to be asymmetric, "peaky" and have tails heavier than those allowed by the normal distribution. Alternative approach to normal distribution gives Student *t* and asymmetric Laplace distributions (*ALD*). Particularly in finance and risk management, the Student *t* distribution has been used instead of the normal distribution, because of its fat tail behaviour, which can be applied to capture financial extreme events. *ALD* is used if we take into account not only fat tails but also skewness of the distributions. Time dependent random process through an *AL* distribution plays an analogous role to Brownian motion (Kozubowski, Podgórski, 2003).

1.1 Asymmetric Laplace distribution

A random variable *X* has an asymmetric Laplace distribution, denoted as $AL(\mu,\sigma,p)$, if there are parameters $\mu \in R, \sigma \geq 0$ such that its characteristic function has the following form

$$\psi(t) = \left[1 + \sigma^2 t^2 - i\mu t\right]^{-1} \tag{2}$$

and probability density function has the form

$$f(x|\mu,\sigma,p) = \frac{p}{\sigma}\exp\left\{-\left(\frac{1}{1-p}I_{[x>\mu]} + \frac{1}{p}I_{[x<\mu]}\right)\frac{k}{\sigma}|x-\mu|\right\}, \tag{3}$$

where μ is the location parameter, σ is a standard deviation of *X*, *p* is the shape parameter taking on values between 0 and 1, $k = k_p = \sqrt{p^2 + (1-p)^2}$.

The shape parameter p controls the skewness and kurtosis of the asymmetric Laplace distribution (Lu, Li, 2011).

The *AL* distribution has several special cases. The distribution is

- degenerate at 0: $\mu = \sigma = 0 \Rightarrow \psi(t) = 1$,
- symmetric Laplace distribution: $\mu = 0, \sigma \neq 0$,
- exponential distribution: $\mu \neq 0, \sigma = 0$.

Random variable can be used for computer simulation of *AL,* for example exponential mixture representation (Kozubowski, Podgórski, 2003):

$$Y_{\mu,\sigma} = \sigma \cdot I_{\kappa} \cdot Z, \tag{4}$$

where I_{κ} is a discrete random variable taking values $-\kappa$ and $1/\kappa$ with probabilities $p = \frac{\kappa^2}{(1+\kappa^2)}$ and $q = \frac{1}{1+\kappa^2}$, respectively, Z is a standard exponential random variable. Note, that we obtain symmetric Laplace distribution for $\mu = 0$ and $\kappa = 1$.

1.2 Skewed *EWMA* forecasting of the volatility

Asymmetric Laplace distribution is a base for skewed–*EWMA* estimator of the volatility of the risk factors and it was developed by Lu, Huang and Gerlach in 2010. This estimator considers both skewness and heavy tails in financial return distribution and it generalizes the robust *EWMA* estimator as a special case.

$$\sigma_{t+1} = (1-\lambda)\sum_{i=0}^{\infty} \lambda^i \left(\frac{k}{1-p} I_{[r_{t-i}>0]} + \frac{k}{p} I_{[r_{t-i}<0]} \right) |r_{t-i}|, \tag{5}$$

$$\hat{\sigma} = \frac{1}{n}\sum_{i=1}^{n} \left(\frac{k}{p} I_{[r_i<0]} + \frac{k}{1-p} I_{[r_i>0]} \right) |r_i| \tag{6}$$

where $k = k(p) = \sqrt{p^2 + (1-p)^2}$ and λ is a decaying factor and r_t are returns of the risk factors at time t.

Through iteration (5) can be expressed as

$$\sigma_{t+1} = \lambda\sigma_t + (1-\lambda)\left(\frac{k}{1-p} I_{[r_t>0]} + \frac{k}{p} I_{[r_t<0]} \right) |r_t| \tag{7}$$

If p=0.5, we obtain symmetric Laplace distribution and (7) reduce to

$$\sigma_{t+1} = \lambda\sigma_t + (1-\lambda)\sqrt{2}|r_t|, \; 0<\lambda<1 \tag{8}$$

$$\hat{\sigma} = \frac{1}{n}\sum_{t=1}^{n} \sqrt{2}|r_t|, \tag{9}$$

which was proposed by (Guermat and Harris, 2001).

If $p \neq 0.5$, then the contribution of the positive/negative value of r_t to the σ_{t+1} is quite different. The case r_t>0 means a good news and the cases r_t<0 means bad news. The volatility of this news is well characterized in (7). As it is written in (Lu, Huang and Gerlach, 2010), this skewed–*EWMA* estimate of standard deviation (8) is a special first order threshold *GARCH* (*TGARCH*) model.

Parameter p is a constant estimated by

$$p = \frac{1}{1+\sqrt{\frac{u}{v}}}, \tag{10}$$

where

$$u = \frac{1}{n}\sum_{i=1}^{n}|r_i| I_{[r_i>0]}, \quad v = \frac{1}{n}\sum_{i=1}^{n}|r_i| I_{[r_i<0]} \tag{11}$$

Obviously, u is the averaged positive return and v is the absolute value of the averaged negative return. The larger u is better; but larger v is worse for investment.

1.3 Skewed *EWMA* VaR with copula

Copulas were initially introduced by (Sklar, 1959). Let H denote a joint distribution of function with margins F_1, F_2, then there exists a unique copula C

$$H(x_1; x_2) = C(F_1(x_1); F_2(x_2)); \tag{12}$$

if F_1; F_2 are continuous functions. The copula model interprets multivariate distributions by coupling the marginal distribution function $F_1(x_1)$; $F_2(x_2)$ with the dependence structure C (Nelsen, 1998, referenced by Zhang &and Ng, 2010). In other words, the joint distribution can be expressed by combining the marginal distributions with the dependence structure, yielding

$$C(u_1, u_2) = H(F_1^{-1}(u_1), F_2^{-1}(u_2)) \tag{13}$$

with $u \in [0,1]^2$, and $F_{i-1}(\bullet)$ denoting the inverse of the marginal distribution $F_i(\bullet)$.

In this paper we use the Student t copula and Clayton copula of the random vector u (Alexander, 2008), (McNeil et al., 2005). Both copulas are important for modelling many relationships between financial asset returns and therefore they are commonly used in market risk analysis. Student t copula captures fat tails and Clayton copula captures the asymmetric lower tail dependence.

This paper was inspired by paper (Lu, Li, 2011), where the authors describe an algorithm for forecasting the VaR of a portfolio by skewed–*EWMA* method and Archimedean copulas. In this paper we extend their algorithm for elliptical copulas (Student t copula). We generate random vector u from asymmetric Laplace distribution for volatility computed using skewed–EWMA method, in our approach. Here we outline this algorithm:

1. We compute historical daily log returns of the two financial assets up to time t.
2. We calibrate both financial time series by Student t distribution using MLE method (Zhang, Ng, 2010).
3. We calibrate Student t copula using EML[1] and IFM[2] method.
4. We calculate the parameters ($\mu_{i,t+1}$, $\sigma_{i,t+1}$, p) in the AL distributions at time t+1, for i=1, 2 using equation (7) and (10).
5. We generate (u,v) from ALD with parameters obtained in the previous step using (4).
6. We calculate simulated value of the portfolio return at time t+1 from Student t copula for (u,v).
7. We repeat steps 5–6 m times and according to the m simulated values of the portfolio return at time t+1 we determine VaR_α using (1).

2 Results and Discussion

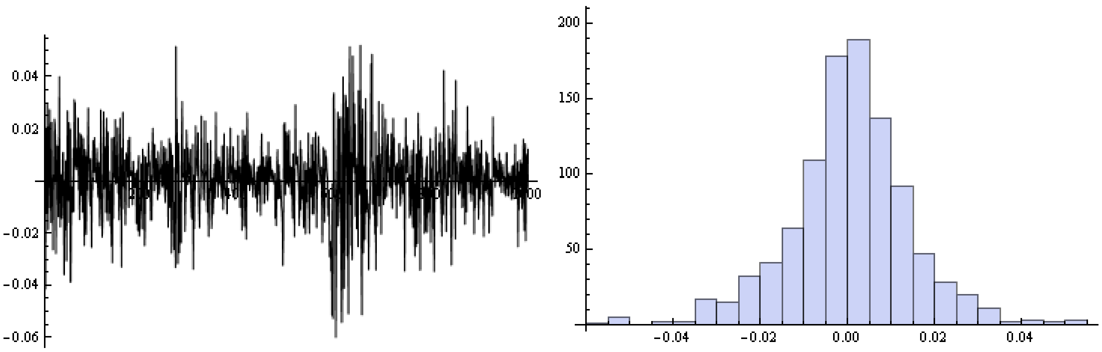
We generate out–of–sample VaR forecast for portfolio which contains two European market indexes: DAX and EUR STOXX 50. Data has been obtained from Bloomberg for period from April 16, 2009 to March 5, 2013. We have analysed 1000 data. All estimation processes were carried out in Wolfram Mathematica v. 9. We compute daily log returns of the closing price for all return series. A sample plot is enough to observe volatility clustering for all return series (Figure 1, Figure 2). Table 1 provides summary statistics of both daily log return series and gives, for example, the mean, standard deviation, skewness, kurtosis, Jarque–Bera test of normality. In both cases, the null hypothesis of normality is rejected at any level of significance, and there is an evidence of significant excess kurtosis of the return series. This indicates that the distributions of these return series are non–Gaussian (Figure 1, Figure 2).

Dependence between the two risk factors has been described by Pearson correlation coefficient $\rho_{Pearson}$. The Kendall's correlation coefficient ρ_{Kendal} has been estimated in order to calculate the parameter θ of the Clayton copula. The Spearman correlation coefficient $\rho_{Spearman}$ has been used for IMF estimating the parameters of the Student t copula. We have obtained parameter v using the maximum likelihood method. We have obtained the correlation

[1] The EML approach estimates the parameters ν and ρ of the Student t copula simultaneously
[2] The IFM approach estimates the parameters ν and ρ of the Student t copula separately

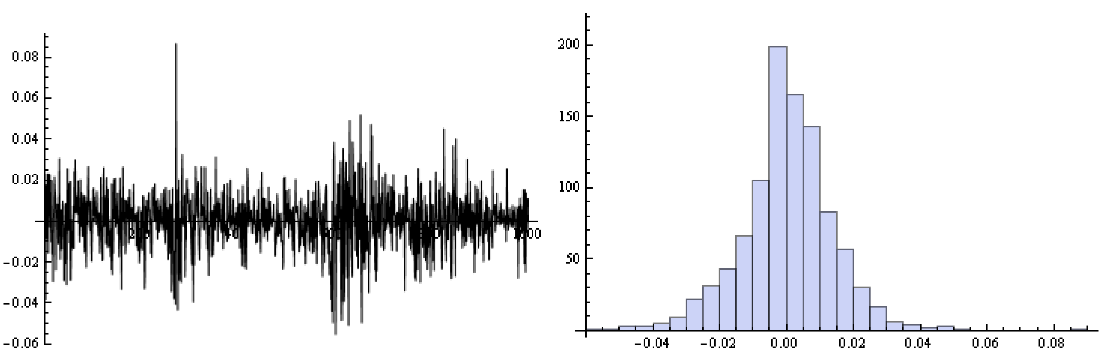
coefficient $\rho_{MLE\text{-}estim}$ again using the maximum likelihood method, but we have simultaneously estimated v and $\rho_{MLE\text{-}estim}$ (Tab. 2).

Fig. 1: DAX log returns and histogram based on the daily data (04/16/09 - 3/5/13)



Source: Calculated by authors with Wolfram Mathematica software based on data from Bloomberg

Fig. 2: EUR STOXX 50 log returns and histogram based on the daily data (04/16/09 - 3/5/13)



Source: Calculated by the authors with Wolfram Mathematica software based on data from Bloomberg

Tab. 1: Descriptive statistics based on the daily log return series, 04/16/09 - 3/5/13

Descriptive statistics	DAX	EUR STOXX 50
Mean	0.001	0.000
Mean Ann.	0.131	0.057
Median	0.001	0.000
Max	0.052	0.087
Min	-0.060	-0.055
Stand. Dev.	0.014	0.014
Stand. Dev. Ann	0.222	0.222
Skewness	-0.202	-0.026
Kurtosis	4.840	5.543
Jarque-Bera *t* stat	150.931	274.923
Jarque-Bera *p* value	0	0
Sample size	1000	1000

Source: Calculated by the authors with Wolfram Mathematica software based on data from Bloomberg

Tab. 2: Estimated correlation coefficients (Pearson, Kendal, Spearman and MLE estimate) based on the daily log return series, 04/16/09-3/5/13

$\rho_{Pearson}$	ρ_{Kendal}	$\rho_{Spearman}$	$\rho_{MLE\text{-}estim}$
0.959	0.958	0.954	0.971

Source: Calculated by the authors with Wolfram Mathematica software based on data from Bloomberg

The degree of freedom has been obtained for standardized returns (with zero mean and unit variance) using maximum likelihood to fit a standardized *t* distribution to both standardized log returns series. These calibrated parameters are shown in Table 3.

Tab. 3: Estimated VaR using Student *t* copula and Student *t* marginals

***t*- marginals** t_{DAX}**=4.406** $t_{EURSTOXX}$**=4.852**	**Student *t*- copula** ν **=6.681;**ρ_S**=0.954 IMF estimate**	**Student *t*- copula** ν **=6.654;**ρ**=0.971 EML estimate**	**Clayton copula** θ**=8.806**
1% 10–day VaR	0.925%	0.932%	0.631%
5% 10–day VaR	0.528%	0.531%	0.442%

Source: Calculated by the authors with Wolfram Mathematica software based on data from Bloomberg

Tab. 4: Estimated VaR using ALD Student *t* copula and Student *t* marginals

Skewed EWMA σ_{DAX}**=13.992%** $\sigma_{EURSTOXX}$**=14.171%**	**Student *t*- copula** ν **=6.681;**ρ**=0.954**	**Student *t*- copula** ν **=6.654;**ρ**=0.971**	**Clayton copula** θ **=8.806**
1% 10–day VaR	0.961%	0.945%	0.623%
5% 10–day VaR	0.626%	0.617%	0.471%

Source: Calculated by the authors with Wolfram Mathematica software based on data from Bloomberg

For the purposes of this article, we created a portfolio composed of indexes DAX and EUR STOXX 50 in ratio 6:4. VaR estimates obtained by different approaches are specified in Tables 3 and 4. The Table 3 shows *VaR* estimates obtained by classical approach. These estimations are based on a random vector *u*, which has a uniform distribution. We used asymmetric Laplace distribution for generating future potential returns in Table 4. As a result of considering the skewness and fat-tail of the risk factor distribution, estimations of VaR are higher than those which we obtain by using Student *t* distribution.

Conclusion

In this paper we propose a skewed–*EWMA* forecasting of VaR_α using Student *t* copula and Clayton copula and Monte Carlo simulations. Empirical application shows that if we take into account the asymmetry of the risk factor distributions then we estimate VaR_{α} of the portfolio more precisely.

The key to a meaningful implementation of Monte Carlo simulation is making reliable judgements on which statistical distribution is appropriate for which risk factors and estimating the parameters of the selected distributions. In practice, a wide array of distributions can be used for different risk factors.

Acknowledgment

The work on this paper has been supported by VEGA grant agency, grant number 1/0279/11.

References

Alexander, C. 2008. Market Risk Analysis. Chichester, England: John Wiley & Sons, Chichester.

Bohdalová, M. 2007. A comparison of Value–at–Risk methods for measurement of the financial risk. E-Leader International Journal, Volume 2, Number 2, http://www.g-casa.com, ISSN 1935-4819, Chinese American Scholars Association, New York, New York, USA, June.

Dempster, M., Howard, A. 2002. Risk Management Value at Risk and beyond. Cambridge, England: Cambridge University Press, Cambrige.

Dowd, K. 2002. Measuring Market Risk. Chichester, England: John Wiley & Sons, Chichester.

Guermat, C., Harris, R. D. F. 2001. Robust Conditional Variance Estimation and Value-at-Risk. Journal of Risk, 4, 25-41. Available at SSRN: http://dx.doi.org/10.2139/ssrn.254569.

Embrechts, P., Lindskog, F.,McNeil, A. J. 2001. Modelling Dependence with Copulas and Applications to Risk Management. Zürich, 2001, Available at: http://www.math.ethz.ch/finance

Jorion, P. 2009. Financial Risk Manager Handbook. Chichester, England: John Wiley & Sons, Chichester.

Kotz, S., Kozubowski, T. J., Podgórski, K. 2003. An Asymmetric Multivariate Laplace Distribution. Available at: http://wolfweb.unr.edu/homepage/tkozubow/0_alm.pdf

Kotz, S., Kozubowski, T. J., Podgórski, K. 2001. The Laplace Distribution and generalizations. A Revisit with Applications to Communications, Economics, Engineering and Finance. Birkhäuser Boston

Kozubowski, T. J., Podgórski, K. 2003. A Class of Asymmetric Distributions. Available at: http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.44.2654

Lu, Z., Huang, H., Gerlach, R. 2010. Estimating Value at Risk, From JP Morgan`s Standard–EWMA to skewed-EWMA Forecasting ; OME Working Paper No: 01/2010, Faculty of Economics and Business, The University of Sydney

Lu, Z., Li, S. 2011. Estimating Value–at–Risk for Portfolios: Skewed–EWMA Forecasting via Copula. Australian actuarial journal, Vol. 17, Issue 1, pp.87–115

McNeil, A. J., Frey, R., Embrechts, P. 2005. Quantitative Risk Management. Concepts, Techniques and Tools. Princeton University Press. Princeton and Oxford.

Nelsen, R. B. 1998. An Introduction to Copulas. Springer, Berlin, Germany.

Zhang, J., Ng, W. L. 2010. *EML*–estimation of multivariate *t*–copulas with heuristic optimization. In Proceedings of International Conference on Computer Science and Information Technology, pp. 469-473.

Contact

Mária Bohdalová
Comenius University in Bratislava
Faculty of Management
Odbojárov 10
820 05 Bratislava 25
maria.bohdalova@fm.uniba.sk

Michal Greguš
Comenius University in Bratislava
Faculty of Management
Odbojárov 10
820 05 Bratislava 25
michal.gregus@fm.uniba.sk