BETA DISTRIBUTED CREDIT SCORE - ESTIMATION OF ITS J-DIVERGENCE

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Abstract

It is known, that Beta distribution could provide a reasonable approximation of distribution of a credit scores, which are the outcome of credit scoring models, i.e. models which are used to determine the probability of default (i.e. when the client fails to meet his or her credit obligations). Credit scoring models are used in practice in the majority of decisions related to the granting of credits, and are thus inherently part of the majority of processes (approval, enforcement, commercial, etc.) in the financial sector. Besides Gini coefficient or K-S statistic, J-divergence (also called Information value) is widely used to assess discriminatory power of credit scoring models. However, empirical estimator using deciles of scores, which is the common way how to compute the J-divergence, may lead to strongly biased results. The main aim of this paper is to describe properties of alternative, both parametric and non-parametric, estimators of J-divergence for credit scoring models with Beta distributed scores. As we show, the parametric and ESIS estimators are much more appropriate to use considering both the bias and mean squared error. Indeed, better estimator leads to better assessment of models, what may lead to better credit scoring models used in practice.

Key words: J-divergence, Information Value, Credit Scoring, Beta Distribution.

JEL Code: E51, C14, C63

Introduction

J-divergence is one of the frequently used ways of describing the difference between two probability distributions. When considering the quality (discriminatory power) of a classification model, maximally different conditional probability distributions is exactly what one aims to. Thus the J-divergence is very suitable, and also widely used, to assess the quality of classification models. It is also known under the name of Information value, in the case of its use for the purpose of scoring models, e.g. credit scoring models that are used to determine the probability of default (i.e. when the client fails to meet his or her credit obligations). Credit scoring models are used in practice in the majority of decisions relating to the granting of loans, and are inherently part of the processes (approval, collection, sales, etc.) in the financial sector. Development methodology of credit scoring models and methods for assessing their quality can be found in articles such as Hand and Henley (1997), Thomas (2000) Vojtek and Kočenda (2006) or Crook et al. (2007) and books like Anderson (2007) or Thomas (2009).

This paper deals primarily with the J-divergence, which is one of the widely used indices (next to the Gini index and KS statistics, see Crook et al. (2007) or Řezáč and Řezáč (2011) for details) for assessment of the quality of credit scoring models. Usually it is calculated by a discretization of the score into intervals using deciles with the requirement for a nonzero number of observations in all intervals. However, this could lead to strongly biased estimate of the J-divergence. As an alternative method to the empirical estimates one can use the kernel smoothing theory, which allows to estimate unknown densities and consequently, using some numerical method for integration, to estimate value of the J-divergence. Another alternative is the empirical estimates with supervised interval selection (ESIS) proposed and discussed in Řezáč (2011). Details connected to the kernel estimates and a discussion concerning both these approaches one can find there as well.

The main objective of this paper is to describe the behaviour of the J-divergence estimates of credit scoring models with Beta distributed score. The second and the third chapter deal with the methodology of these estimates, including algorithms or reference to the relevant literature. The fourth chapter is then devoted to justify appropriateness of Beta distribution on a real data, though the choice of this distribution type was justified in the literature, e.g. in Jankowitsch et al. (2007) or Moraux (2010). Furthermore, this chapter is devoted to computation of the J-divergence on the real data. There are also, using a simulation study, discussed the properties of estimates described within the paper.

1 J-divergence for Beta distributed score

Jeffrey divergence (J-divergence) of two random variables X_0 and X_1 with densities $f_0(x)$ and $f_1(x)$ is defined as symmetrised Kullback-Leibler divergence, i.e.

$$D_J(X_0, X_1) = D_{KL}(X_0 : X_1) + D_{KL}(X_1 : X_0) = \int_{-\infty}^{\infty} \left(f_0(x) - f_1(x)\right) \ln\left(\frac{f_0(x)}{f_1(x)}\right) dx, \tag{1}$$

where Kullback-Leibler divergence $D_{KL}(X_0 : X_1)$ is given by

$$D_{KL}(X_0 : X_1) = \int_{-\infty}^{\infty} \left(f_0(x)\right) \ln\left(\frac{f_0(x)}{f_1(x)}\right) dx. \tag{2}$$

Consider thus two random variables X_0 and X_1 representing suitably transformed outputs of the credit scoring model for bad (clients in default) and good clients. Let these random variables be Beta distributed with densities $f_0(x)$ and $f_1(x)$ defined by

$$f_0(x) = \begin{cases} \dfrac{1}{B(\alpha_0, \beta_0) \cdot \sigma_0^{\alpha_0 + \beta_0 - 1}} (x - \vartheta_0)^{\alpha_0 - 1} \cdot (\sigma_0 + \vartheta_0 - x)^{\beta_0 - 1} & \text{for } \vartheta_0 < x < \vartheta_0 + \sigma_0 \\ 0 & \text{for } x \le \vartheta_0 \text{ or } x \ge \vartheta_0 + \sigma_0 \end{cases} \tag{3}$$

$$f_1(x) = \begin{cases} \dfrac{1}{B(\alpha_1, \beta_1) \cdot \sigma_1^{\alpha_1 + \beta_1 - 1}} (x - \vartheta_1)^{\alpha_1 - 1} \cdot (\sigma_1 + \vartheta_1 - x)^{\beta_1 - 1} & \text{for } \vartheta_1 < x < \vartheta_1 + \sigma_1 \\ 0 & \text{for } x \le \vartheta_1 \text{ or } x \ge \vartheta_1 + \sigma_1. \end{cases} \tag{4}$$

It is easy to show that the transformations $y = \dfrac{x - \vartheta_i}{\sigma_i}$, i = 0,1, convert random variables X_0 and X_1 to random variables Y_0 a Y_1 with densities

$$g_0(y) = \begin{cases} \dfrac{1}{B(\alpha_0, \beta_0)} y^{\alpha_0 - 1} \cdot (1 - y)^{\beta_0 - 1} & \text{for } 0 < y < 1 \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

$$g_1(y) = \begin{cases} \dfrac{1}{B(\alpha_1, \beta_1)} y^{\alpha_1 - 1} \cdot (1 - y)^{\beta_1 - 1} & \text{for } 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

For such distributed random variables one can find analytical expression of the Kullback-Leibler divergence, and hence also of the J-divergence. We get therefore

$$\begin{aligned} D_J(Y_0, Y_1) = {} & (\alpha_1 - \alpha_0) \cdot \left(\psi(\alpha_1) - \psi(\alpha_0)\right) + (\beta_1 - \beta_0) \cdot \left(\psi(\beta_1) - \psi(\beta_0)\right) + \\ & + (\alpha_1 - \alpha_0 + \beta_1 - \beta_0) \cdot \left(\psi(\alpha_0 + \beta_0) - \psi(\alpha_1 + \beta_1)\right), \end{aligned} \tag{7}$$

where $\psi(t)$ is the digamma function. See Gradshtein and Ryzhik (1965) or Medina and Moll (2009) for details about the digamma function. The J-divergence can also be calculated by approximative formula using the relationship $\psi(t) \approx \ln(t - 0{,}5)$, see Johnson, Kotz and Balakrishnan (1995). Then it holds

$$D_J(Y_0, Y_1) \approx \ln\left[\left(\frac{\alpha_1 - 0{,}5}{\alpha_0 - 0{,}5} \right)^{\alpha_1 - \alpha_0} \cdot \left(\frac{\beta_1 - 0{,}5}{\beta_0 - 0{,}5} \right)^{\beta_1 - \beta_0} \cdot \left(\frac{\alpha_0 + \beta_0 - 0{,}5}{\alpha_1 + \beta_1 - 0{,}5} \right)^{\alpha_1 - \alpha_0 + \beta_1 - \beta_0} \right]. \tag{8}$$

Furthermore, one still needs to estimate the parameters $\alpha_0, \alpha_1, \beta_0$ and β_1 for the practical estimation of the J-divergence. Typically, this is done using the MLE estimates. Those are available for example in Univariate procedure in SAS system. Computational schemes of these MLE estimates can then be found in Johnson, Kotz a Balakrishnan (1995). Overall, we obtain the parametric estimates by this procedure.

One point has to be mentioned here. The J-divergence is not generally invariant with respect to transformations. Indeed, this holds for transformations used for converting four-parameters Beta distributed variables given by (3) and (4) to two-parameters Beta distributed variables given by (5) and (6). Nevertheless, when comparing the discriminative power of several credit scoring models on the same data, then this property (disadvantage) does not matter. And what is quite important, estimation of parameters in (3) and (4) and consequent computation of the J-divergence is quite complicated. From this perspective, it seems to be appropriate to use parametric estimate given by (7) or (8).

2 Non-parametric estimates of J-divergence

In practice, the most commonly used non-parametric estimators of the J-divergence are the empirical estimates. These are based on the idea of replacing unknown densities by empirical estimates of these densities, de facto using appropriate relative frequencies. Let's have n_0 score values $s_{0_i}, i = 1, \ldots, n_0$ for bad clients and n_1 score values $s_{1_i}, i = 1, \ldots, n_1$ for good clients and denote L (resp. H) as the minimum (resp. maximum) of all values.

Let's divide the interval $[L, H]$ up to r subintervals $[q_0, q_1], (q_1, q_2], \ldots, (q_{r-1}, q_r]$, where $q_0 = L - 1,\ q_r = H + 1,\ q_i,\ i = 1, \ldots, r-1$ and $q_i,\ i = 1, \ldots, r-1$ are suitable border points, e.g. appropriate quantiles of score of all clients. Set

$$\begin{aligned} n_{0_j} &= \sum_{i=1}^{n_0} I(s_{0_i} \in (q_{j-1}, q_j]) \\ n_{1_j} &= \sum_{i=1}^{n_1} I(s_{1_i} \in (q_{j-1}, q_j]) \quad j = 1, \ldots, r \end{aligned} \tag{9}$$

observed counts of bad or good clients in each interval. Denote $\hat{f}_{IV}(j)$ the contribution to the information value on j^th^ interval, calculated by

$$\hat{f}_{IV}(j) = \left(\frac{n_{1_j}}{n_1} - \frac{n_{0_j}}{n_0} \right) \ln \left(\frac{n_{1_j} n_0}{n_{0_j} n_1} \right),\ j = 1, \ldots, r. \tag{10}$$

The empirical estimate of the J-divergence is given by

$$\hat{D}_J = \sum_{j=1}^{r} \hat{f}_{IV}(j). \tag{11}$$

A special case is the decile estimate which uses scores of all clients and $r = 10$ to determine the boundaries of the interval q_i. Further algorithms are then e.g. ESIS, see Řezáč (2011), ESIS1, see Řezáč and Koláček (2011), or ESIS2, see Řezáč (2012).

Further possible way how to estimate the J-divergence is the usage of the theory of kernel density estimates. For given M+1 equidistant point of the score $L = x_0, x_1, \ldots, x_M = H$ we have

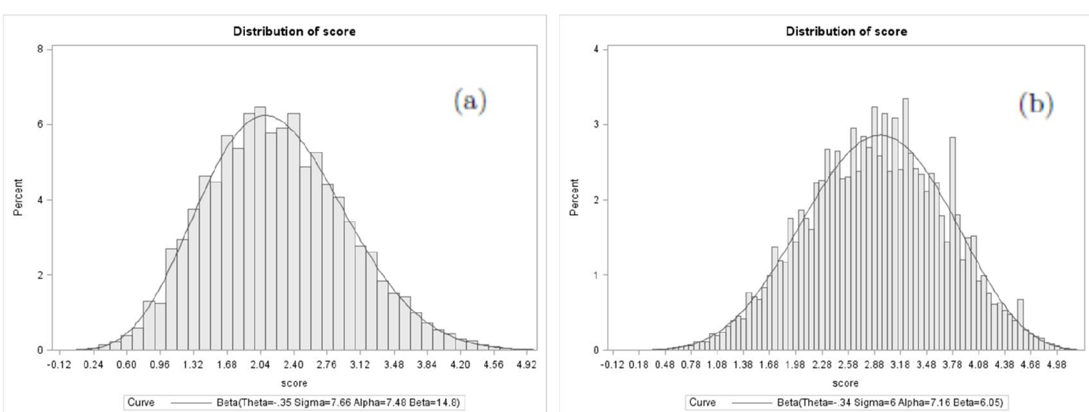
$$\hat{D}_J = \frac{H-L}{2M}\left(\tilde{f}_{IV}(L) + 2\sum_{i=1}^{M-1}\tilde{f}_{IV}(x_i) + \tilde{f}_{IV}(H)\right), \tag{12}$$

where $\tilde{f}_{IV}(x_i)$ are estimated contributions to the J-divergence given by appropriate kernel estimates of the unknown densities of bad and good clients' scores. See Řezáč (2011) for more details.

3 Results

The logical question is why to consider just the Beta distribution. Some arguments were given in Jankowitsch et al. (2007) and Moraux (2010). Furthermore, the answer can be found in the following Figures 1 and 2 and Tables 1 and 2, obtained using the SAS UNIVARIATE procedure. There was some real data provided by a financial institution including output of a credit scoring model (inverse logit transformation of 1 minus the probability of default) and a good/bad indicator of the client. The data range was 176,878 observations, see Řezáč a Řezáč (2011) for more details.

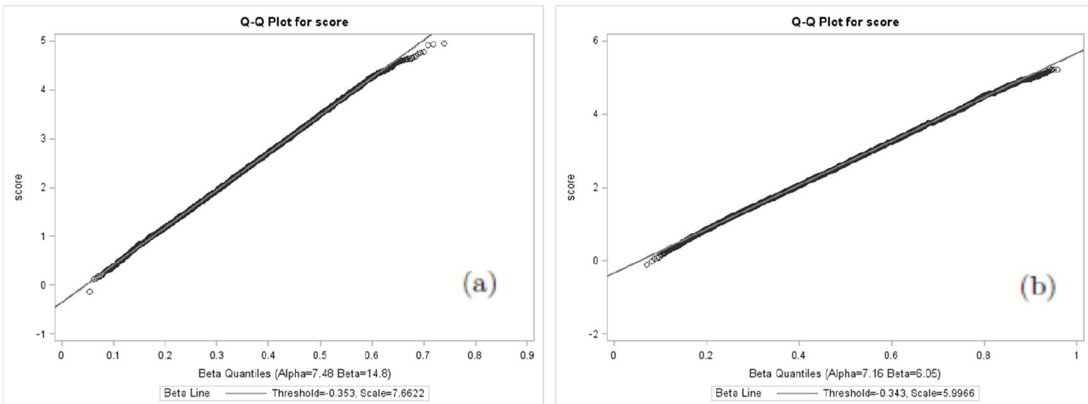
Fig. 1: Fitted Beta distributions of scores. a) for bad, b) for good clients.



Source: Own construction.

From the Figure 1 (fitted Beta densities and histograms) and especially from the Figure 2 (Q-Q charts) it is obvious that the choice of the Beta distribution was appropriate.

Fig. 2: Q-Q charts for fitted Beta distributions of scores. a) for bad, b) for good clients.



Source: Own construction

The following Table 1 contains the results of goodness of fit (GoF) tests of the examined data. In case of bad clients' score, all considered tests did not reject the hypothesis of a Beta distribution. On the other hand, in case of good clients' score there was approximately tenfold growth of test statistics for the Cramer-von Mises and Anderson-Darling tests but the test statistics of the Kolmogorov-Smirnov test remained at approximately the same value. Overall, all three tests rejected the hypothesis of a Beta distributed score in the case of good clients' score.

Tab. 1: Goodness of fit tests of scores. a) for bad, b) for good clients.

Goodness-of-Fit Tests for Beta Distribution				
Test	Statistic		p Value	
Kolmogorov-Smirnov	D	0.00863	Pr > D	0.117
Cramer-von Mises	W-Sq	0.09981	Pr > W-Sq	>0.250
Anderson-Darling	A-Sq	0.73667	Pr > A-Sq	>0.250

(a)

Goodness-of-Fit Tests for Beta Distribution				
Test	Statistic		p Value	
Kolmogorov-Smirnov	D	0.00905	Pr > D	<0.001
Cramer-von Mises	W-Sq	1.05559	Pr > W-Sq	0.002
Anderson-Darling	A-Sq	8.24061	Pr > A-Sq	<0.001

(b)

Source: Own construction.

The problem (results of GoF tests for good clients' score) lies in the very large range (approximately 160,000 observations) of data. It is commonly known that with a large data set, the GoF test becomes very sensitive to even very small, inconsequential departures from a distribution. Due to this phenomenon it is recommended (for large data set) to follow the result of Q-Q chart rather than results of GoF tests. Even if we did not want to follow this recommendation, it applies that when performing the same tests (more specifically, we did one thousand tests) on a random sample of good clients' score comprising 10% of the original number of observations we got approximately the same results as for bad clients' score (even with higher p-values for all three tests). Overall, we stated that scores of bad and good clients could be considered Beta distributed.

Table 2 contains parameters of the fitted Beta distributions. The most striking difference between good and bad clients' score is given by the parameter beta (14.8 vs. 6.05) and the parameter sigma (7.66 vs. 5.99).

Tab. 2: Parameters of Beta distributed scores. a) for bad, b) for good clients.

Parameters for Beta Distribution		
Parameter	Symbol	Estimate
Threshold	Theta	-0.35327
Scale	Sigma	7.662249
Shape	Alpha	7.479128
Shape	Beta	14.81027
Mean		2.217774
Std Dev		0.749697

(a)

Parameters for Beta Distribution		
Parameter	Symbol	Estimate
Threshold	Theta	-0.34301
Scale	Sigma	5.996635
Shape	Alpha	7.15525
Shape	Beta	6.05231
Mean		2.905692
Std Dev		0.792681

(b)

Source: Own construction.

The following Table 3 shows values of D_J estimated by the algorithms mentioned above. The last line contains the parameter estimate given by (7) with MLE estimates of parameters. As the best non-parametric estimate seems to be the value 3.117 given by the algorithm ESIS1.

Tab. 3: Estimates of D_J.

	D_J
decil	2.508551
kern	2.797372
esis	2.945658
esis1	**3.117013**
esis2	2.967163
param	3.403594

Source: Own construction.

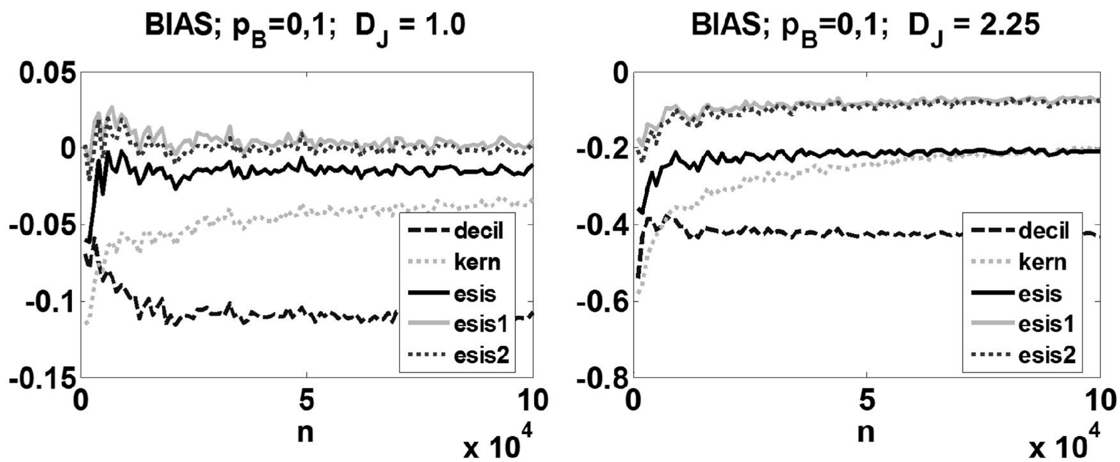
The question, however, is the general behaviour of these algorithms. A very common way to assess the quality / properties of some parameter estimates or statistics are bias (bias) and mean square error (MSE), defined by

$$bias = E\left(\hat{D}_J\right) - D_J, \tag{13}$$

$$MSE = E\left(\left(\hat{D}_J - D\right)^2\right). \tag{14}$$

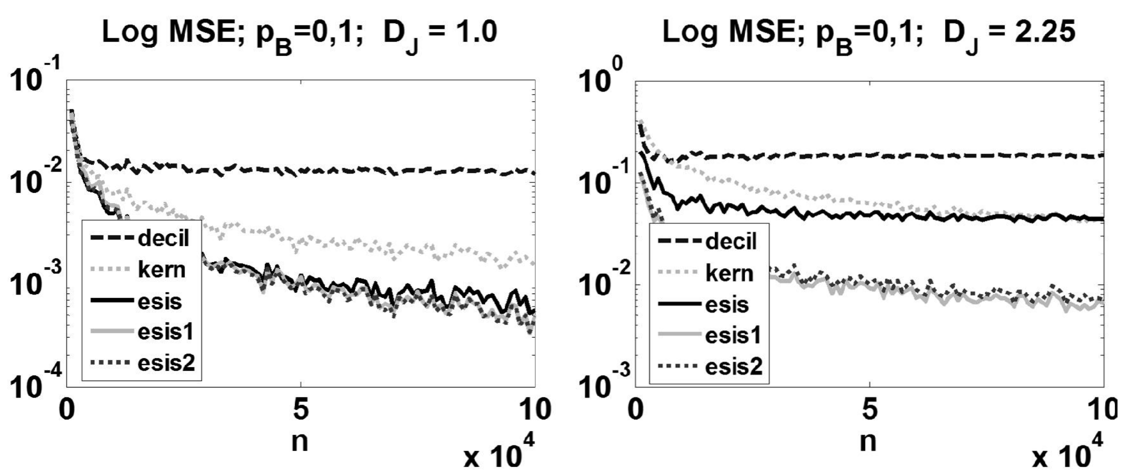
The following Figures 3 and 4 show the properties of these algorithms from this perspective. Simulation study leading to these results was carried out as follows. We consider n clients, $n \cdot p_B$ bad and $n \cdot (1 - p_B)$ good (p_B is the relative frequency of bad clients). In our case we choose p_B = 0.1, which is the closest to the aforementioned real data. Furthermore, we consider the parameters of Beta distribution resulting the value D_J = 1.00 and 2.25. Range of data sample we choose n = 1000 and n = 100 000. First, we generate scores of bad and good clients, depending on the selected parameters. Then we calculate all of the aforementioned estimates. This process was repeated one thousand times. Mean values for the bias and the MSE are then calculated as the arithmetic means.

Fig. 3: Bias of estimates of $\hat{D}_J$ for Beta distributed of scores.



Source: Own construction.

Fig. 4: Logarithm of MSE of estimates of $\hat{D}_J$ for Beta distributed of scores.



Source: Own construction.

From the Figures 3 and 4 it is apparent that the decile estimate is significantly biased, specifically undervalued. The value of log MSE became quite quickly stabilized and with increasing number of observations did not fall. Overall, this estimate is thus not very suitable. In contrast, algorithms ESIS1 and ESIS2 led in the case of a weaker model (D_J = 1.00) to almost unbiased estimate. For a stronger model (D_J = 2.25) are their properties worse. However, they were the best of all considered methods of estimating D_J.

Conclusion

The aim of this paper was to describe properties of the selected estimates of J-divergence (also called Information value) of credit scoring models with Beta distributed scores. It was given a formula for the theoretical value of the J-divergence assuming this type of distribution. Its knowledge enabled the both compute parametric estimates, but also to assess the quality of non-parametric estimates. On real data it was presented an estimate of the J-divergence. In addition, properties of aforementioned estimates have been demonstrated on simulated data from the Beta distribution. Namely, they were the bias and the MSE for the data ranges from 1000 to 100 000. It quite obviously turned out the weaknesses of traditional decile empirical estimation. Conversely, it seemed that the algorithms ESIS1 and ESIS2 were good estimates of J-divergence with Beta distributed score.

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