A MATHEMATICAL PROPERTY OF THE HARMONIC MEAN

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Abstract:
The harmonic mean is one of the important measures of central tendency, which is applied more frequently in averaging speeds when the distances for which each speed is applicable are the same .The harmonic mean is based on reciprocals of the numbers averaged .It is the reciprocal of the arithmetic mean of the reciprocal of the numbers averaged .Experiment shows that whenever we average a group of values the arithmetic mean will be larger than the geometric mean ,and the later will be larger than the harmonic mean (unless all the values averaged are of the same size ,in which case the three averages will be identical) , Properties of the harmonic mean are mentioned in many elementary statistical books. In this paper, the author will give a proof concerning the adequacy of the harmonic mean to mathematical manipulation, that is because, the author has not seen this proof in any of the books or papers.

Key words: Central tendency, Harmonic mean, Mathematical property.
JEL Code: CO1, CO2

Introduction

The harmonic mean is one of the measures of Central tendency, which is useful when averaging rates, and it is desired to keep constant in the average the factor that is variable in the rate.

Some properties of the harmonic mean are mentioned in many elementary statistical books.

In this paper, the author will add another property concerning the susceptibility of the harmonic mean to mathematical operations; this property is stated by following theorem.

Theorem:

(a) if a set of n_1 variates , say , $x_1, x_2, ..., x_{n_1}$ and a set of n_2 variates , say , $y_1, y_2, ..., y_{n_2}$ with harmonic means H_1 and H_2 respectively , then the combined harmonic mean of the two sets of variates ,say , Hc_2 is given by ;

$$Hc_2 = \frac{(n_1 + n_2)H_1 H_2}{n_1 H_2 + n_2 H_1}$$

(b) If m sets of variates , say , $n_1, n_2, \ldots. n_m$ with harmonic means $H_1, H_2, \ldots. H_m$ respectively, then the combined harmonic mean of the m sets of variates H_{cm} is given by ;

$$H_{cm} = \frac{(\sum_{i=1}^{m} n_i)(\prod_{i=1}^{m} H_i)}{\sum_{i=1}^{m} n_i (\prod_{j \neq i} H_j)}; m \geq 2$$

Where

$$\prod_{i=1}^{m} H_i = H_1 H_2 \ldots\ldots H_m$$

Proof:

The harmonic mean of the n set of Variates $x_1, x_2, \ldots., x_n$ is defined as:

$$H = \frac{n}{\sum_{c=1}^{n} \frac{1}{x_c}}$$

(a) For the proof of part (a); consider the n_1 set of variates $x_1, x_2, \ldots\ldots, x_{n_1}$ and the n_2 set of variates $y_1, y_2, \ldots., y_{n_2}$. Let H_1 and H_2 denote the harmonic means of the two sets of variates respectively, then by definition we have;

$$\begin{cases} H_1 = \dfrac{n_1}{\sum_{i=1}^{n1} \frac{1}{x_i}} \\ H_2 = \dfrac{n_2}{\sum_{i=1}^{n_2} \frac{1}{y_i}} \end{cases} \qquad \ldots..(1)$$

And hence we can write

$$\begin{cases} \sum_{i=1}^{n_1} \frac{1}{x_i} = \frac{n_1}{H_1} \\ \sum_{i=1}^{n_2} \frac{1}{y_i} = \frac{n_2}{H_2} \end{cases} \qquad .....(2)$$

Now Let Hc_2 denotes the harmonic mean of the combined two sets of variates n_1 and n_2, Then by definition , we have:

$$H_{C2} = \frac{1}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \ldots + \frac{1}{x_{n_1}} + \frac{1}{y_1} + \frac{1}{y_2} + \ldots. + \frac{1}{y_{n_2}}}$$

From (2) ,and after few steps, we can write:

$$H_{C2} = \frac{n_1 + n_2}{\frac{n_1}{H_1} + \frac{n_2}{H_2}} = \frac{(n_1 + n_2)H_1 H_2}{(n_1 H_2 + n_2 H_1)}$$

Or

$$H_{C2} = \frac{\left(\sum_{i=1}^{2} n_i\right)\left(\prod_{i=1}^{2} H_i\right)}{\sum_{i=1}^{2} n_i \prod_{j \neq i}^{2} H_j} \qquad ....(3)$$

Which completes the proof for part(a).

(b) Suppose we have "*m*" sets of variates $n_1, n_2, \ldots,$ and n_m with harmonic means $H_1, H_2, \ldots.., H_m$ respectively, Let Hc_m be the harmonic mean for the combined *n* sets of variates .For $m=2$. the proof is given in (a).Now let us consider the case when $m=3$, that is we have three sets of variates n_1, n_2, *and* n_3 which are ($x_1, x_2, \ldots\ldots x_{n_1}$),($y_1, y_2, \ldots., y_{n_2}$) *and* ($z_1, z_2, \ldots., z_{n_3}$) with harmonic means H_1, H_2 and H_3 respectively, then the harmonic mean of the combined. three sets of variates is:

$$H_{C3} = \frac{n_1 + n_2 + n_3}{\frac{1}{x_1} + \ldots + \frac{1}{x_{n_1}} + \frac{1}{y_1} + \ldots. + \frac{1}{y_{n_2}} + \frac{1}{z_1} + \ldots + \frac{1}{z_{n_3}}}$$

$$= \frac{n_1 + n_2 + n_3}{\sum_{i=1}^{n_1} \frac{1}{x_i} + \sum_{i=1}^{n_2} \frac{1}{y_i} + \sum_{i=1}^{n_3} \frac{1}{z_i}}$$

From (2), and after few steps we can write:

$$H_{C3} = \frac{\left(\sum_{i=1}^{3} n_i\right)\left(\prod_{i=1}^{3} H_i\right)}{\sum_{i=1}^{3} n_i \quad \left(\prod_{j \neq i}^{3} H_i\right)}$$

If we repeat the same procedure for *m=4,5,…,etc*, then we finally obtain the Harmonic mean of the combined m sets of Variates $n_1, n_2, \ldots, n_m$ with the harmonic means $H_1, H_2, \ldots, H_m$ respectively, say, H_{cm} is as follows:

$$H_{cm} = \frac{n_1 + n_2 + \ldots.. + n_m}{\sum_{i=1}^{n_1} \frac{1}{x_i} + \sum_{i=1}^{n_2} \frac{1}{y_i} + \ldots\ldots. + \sum_{i=1}^{m_3} \frac{1}{z_i}}$$

Again from (2), and after few steps we get:

$$H_{cm} = \frac{(n_1 + n_2 + \ldots\ldots + n_m)(H_1 H_2 \ldots.. H_m)}{n_1 H_2 H_3 \ldots\ldots H_m + \ldots\ldots\ldots. + n_m H_1 \ldots\ldots.. H_{m-1}}$$

$$= \frac{\left(\sum_{i=1}^{m} n_i\right)\left(\prod_{i=1}^{m} H_i\right)}{\sum_{i=1}^{m} n_i \quad \left(\prod_{j \neq i}^{m} H_i\right)}$$

Which completes the proof.

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