# GEOMETRICAL DESCRIPTION OF ONE SURFACE IN ECONOMY

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**Abstract**

The principal object of this paper is the regular parametric surface $M$ in $R^3$ defined by the formula $x(u,v) = (u, v, u^3 - 3uv^2)$. The geometrical description methods we are going to use are based on Cartan's moving frame method and on Weingarten map. The studied map $x(u,v) = (u,v,u^3 - 3uv^2)$ is regular . Let $x: U \subset R^2 \to M \subset R^3$, $(u,v) \in U \subset R^2$, where $U$ is an open neighborhood of the point *q= (u,v)* and $p = x(u,v) \in V \cap M \subset R^3$, where $V$ is an open neighborhood in $R^3$ of the point $x(u,v)$.

**Key words:** Cartan's moving frame method, Weingarten map, Gaussian curvature, Main curvature, moving frame

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## Introduction

Let $U \subset R^2$ is an open neighborhood of a point $(u,v) \in U$ and $x: U \to R^3$ is a regular map (which means that the rank of Jacobian matrix $J(x)(u,v) = 2$ ). A subset $M \subset R^3$ is called regular two dimensional surface in $R^3$ if for each $x = x(u,v)$ there exist an open neighborhood $V$ of $x(u,v) \in R^3$ and the map $x: U \subset R^2 \to M \cap V$ of an open subset $U \subset R^2$ onto $M \cap V$ . The map $x$ is given by the formula

$$x(u,v) = (u,v,u^3 - 3uv^2) .$$

Now we have to construct the moving frame and orthonormal moving frame which is the base for Cartan method. The next method is based on Weingarten mapping.

## 1 Cartan´s method

The moving frame has the form

$$x_u = (1, 0, 3u^2 - 3v^2), \qquad x_v = (0, 1, -6uv), \qquad n = (-3u^2 + 3v^2, 6uv, 1).$$

Symbols $x_u$ and $x_v$ are used instead of $\partial_u x$, $\partial_v x$ etc. Vectors $x_u$ and $x_v$ form the basis of the tangent space $T_{x(u,v)}M$ . On $T_x(M)$ we can construct moving frame ( $x_u$, $x_v$, $x_u \times x_v$ ). As the vectors $x_u$ and $x_v$ are tangent vectors of $M$ the $x_u \times x_v = n$ is a normal vector. Vector $N = \frac{x_u \times x_v}{\|x_u \times x_v\|}$ is the unit normal vector.

## Orthonormal moving frame has the form

$$E_1 = \left( \frac{1}{\sqrt{1+9(u^2-v^2)^2}}, \quad 0, \quad \frac{3(u^2-v^2)}{\sqrt{1+9(u^2-v^2)^2}} \right),$$

$$E_2 = \begin{pmatrix} \dfrac{18\, uv(u^2-v^2)}{\sqrt{1+9(u^2-v^2)^2}\sqrt{1+9(u^2+v^2)^2}}, \\ \dfrac{1+9(u^2-v^2)^2}{\sqrt{1+9(u^2-v^2)^2}\sqrt{1+9(u^2+v^2)^2}}, \\ \dfrac{-6uv}{\sqrt{1+9(u^2-v^2)^2}\sqrt{1+9(u^2+v^2)^2}} \end{pmatrix},$$

$$E_3 = \left( \frac{-3u^2 + 3v^2}{\sqrt{1+9(u^2+v^2)^2}}, \frac{6uv}{\sqrt{1+9(u^2+v^2)^2}}, \frac{1}{\sqrt{1+9(u^2+v^2)^2}} \right),$$

Differential $dE_1$ equals

$$dE_1 = \left( \frac{-18u(u^2-v^2)}{\left[1+9(u^2-v^2)^2\right]^{\frac{3}{2}}}, \quad 0, \quad \frac{6u}{\left[1+9(u^2-v^2)^2\right]^{\frac{3}{2}}} \right) du +$$

$$+ \left( \frac{18v(u^2-v^2)}{\left[1+9(u^2-v^2)^2\right]^{\frac{3}{2}}}, \quad 0, \quad \frac{-6v}{\left[1+9(u^2-v^2)^2\right]^{\frac{3}{2}}} \right) dv.$$

Differential form $\omega_{12}$ is:

$$\omega_{12} = d\,E_1 \cdot E_2 =$$

$$= \frac{-18^2 \cdot u^2 \cdot v \cdot \left(u^2 - v^2\right)^2 - 36 \cdot u^2 \cdot v}{\left(1 + 9 \cdot \left(u^2 - v^2\right)^2\right)^2 \sqrt{1 + 9 \cdot \left(u^2 + v^2\right)^2}}\, du + \frac{18^2 \cdot u \cdot v^2 \cdot \left(u^2 - v^2\right)^2 + 36 \cdot u \cdot v^2}{\left(1 + 9 \cdot \left(u^2 - v^2\right)^2\right)^2 \sqrt{1 + 9 \cdot \left(u^2 + v^2\right)^2}}\, dv =$$

$$= \frac{-36 \cdot u^2 \cdot v \cdot \left[1 + 9 \cdot \left(u^2 - v^2\right)^2\right]}{\left(1 + 9 \cdot \left(u^2 - v^2\right)^2\right)^2 \sqrt{1 + 9 \cdot \left(u^2 + v^2\right)^2}}\, du + \frac{36 \cdot u \cdot v^2 \cdot \left[1 + 9 \cdot \left(u^2 - v^2\right)^2\right]}{\left(1 + 9 \cdot \left(u^2 - v^2\right)^2\right)^2 \sqrt{1 + 9 \cdot \left(u^2 + v^2\right)^2}}\, dv =$$

$$= \frac{-36 \cdot u^2 \cdot v \;\; du + 36 \cdot u \cdot v^2 \;\; dv}{\left[1 + 9 \cdot \left(u^2 - v^2\right)^2\right] \sqrt{1 + 9 \cdot \left(u^2 + v^2\right)^2}} \,.$$

We have

$$\partial_u E_3 = \left[ \frac{-6u\left[1 + 18v^2\left(u^2 + v^2\right)\right]}{\left[1 + 9\left(u^2 + v^2\right)^2\right]^{\frac{3}{2}}}, \frac{6v\left[1 - 9\left(u^4 - v^4\right)\right]}{\left[1 + 9\left(u^2 + v^2\right)^2\right]^{\frac{3}{2}}}, \frac{-18u(u^2 + v^2)}{\left[1 + 9\left(u^2 + v^2\right)^2\right]^{\frac{3}{2}}} \right],$$

$$\partial_u E_3 \cdot E_1 = \frac{-6u\left[1 + 18v^2(u^2 + v^2)\right] - 54u\left(u^2 - v^2\right)\left(u^2 + v^2\right)}{\sqrt{1 + 9\left(u^2 - v^2\right)^2}\,\left[1 + 9\left(u^2 + v^2\right)^2\right]^{\frac{3}{2}}} =$$

$$= \frac{-6u\left[1 + 18u^2v^2 + 18v^4 + 9u^4 - 9v^4\right]}{\sqrt{1 + 9\left(u^2 - v^2\right)^2}\,\left[1 + 9\left(u^2 + v^2\right)^2\right]^{\frac{3}{2}}} =$$

$$= \frac{-6u\left[1 + 9\left(u^2 + v^2\right)^2\right]}{\sqrt{1 + 9\left(u^2 - v^2\right)^2}\,\left[1 + 9\left(u^2 + v^2\right)^2\right]^{\frac{3}{2}}}$$

$$= \frac{-6u}{\sqrt{1 + 9\left(u^2 - v^2\right)^2}\,\sqrt{1 + 9\left(u^2 + v^2\right)^2}} \,,$$

$$\partial_v E_3 = \left\{ \frac{6v\left[1+18u^2(u^2+v^2)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}, \frac{6u\left[1+9(u^4-v^4)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}, \frac{-18v(u^2+v^2)}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}} \right\},$$

$$\partial_v E_3 \cdot E_1 = \frac{6v\left[1+9\left(u^2+v^2\right)^2\right]}{\sqrt{1+9\left(u^2-v^2\right)^2}\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}} = \frac{6v}{\sqrt{1+9\left(u^2-v^2\right)^2}\sqrt{1+9\left(u^2+v^2\right)^2}}.$$

The differential form $\omega_{31}$ is

$$\omega_{31} = dE_3 \cdot E_1 = \frac{-6u\,du + 6v\,dv}{\sqrt{1+9\left(u^2-v^2\right)^2}\ \sqrt{1+9\left(u^2+v^2\right)^2}}.$$

Further we try to construct differential form $\omega_{32}$.

$$\partial_u E_3 \cdot E_2 = \frac{6v\left[1-9\left(u^4-v^4\right)\right]}{\sqrt{1+9\left(u^2-v^2\right)^2}\left(1+9\left(u^2+v^2\right)^2\right)}$$

$$\partial_v E_3 \cdot E_2 = \frac{6u\left[1+9\left(u^4-v^4\right)\right]}{\sqrt{1+9\left(u^2-v^2\right)}\left[1+9\left(u^2+v^2\right)^2\right]}$$

Differential form $\omega_{32}$ is

$$\omega_{32} = \partial_u E_3 \cdot E_2 + \partial_v E_3 \cdot E_2 = \frac{6v\left[1-9\left(u^4-v^4\right)\right]du + 6u\left[1+9\left(u^4-v^4\right)\right]dv}{\sqrt{1+9\left(u^2-v^2\right)^2}\left(1+9\left(u^2+v^2\right)^2\right)}$$

Now we will construct forms $\theta_1$ and $\theta_2$.

$$\theta_1 = E_1 dx = E_1 x_u du + E_1 x_v dv$$

$$E_1 x_u du + E_1 x_v dv = \sqrt{1+9\left(u^2-v^2\right)^2}\,du - \frac{18uv(u^2-v^2)}{\sqrt{1+9\left(u^2-v^2\right)^2}}\,dv.$$

$$E_2 x_u du = \frac{18uv\left(u^2 - v^2\right) - 18uv(u^2 - v^2)}{\sqrt{1 + 9\left(u^2 - v^2\right)^2}\sqrt{1 + 9\left(u^2 + v^2\right)^2}} du = 0,$$

$$E_2 x_v dv = \frac{1 + 9\left(u^2 - v^2\right)^2 + 36u^2v^2}{\sqrt{1 + 9\left(u^2 - v^2\right)^2}\ \sqrt{1 + 9\left(u^2 + v^2\right)^2}} dv = \frac{1 + 9\left(u^2 + v^2\right)^2}{\sqrt{1 + 9\left(u^2 - v^2\right)^2}\ \sqrt{1 + 9\left(u^2 + v^2\right)^2}} dv.$$

The exterior product of forms $\theta_1$ and $\theta_2$ is

$$\theta_1 \wedge \theta_2 = \frac{1 + 9\left(u^2 + v^2\right)^2}{\sqrt{1 + 9\left(u^2 + v^2\right)^2}} du \wedge dv = \sqrt{1 + 9\left(u^2 + v^2\right)^2}\, du \wedge dv,$$

from which follows

$$du \wedge dv = \frac{1}{\sqrt{1 + 9\left(u^2 + v^2\right)^2}} \theta_1 \wedge \theta_2.$$

For control we have

$$dE_1 = \left( \frac{-18u\left(u^2 - v^2\right)}{\left[1 + 9\left(u^2 - v^2\right)^2\right]^{\frac{3}{2}}},\quad 0,\quad \frac{6u}{\left[1 + 9\left(u^2 - v^2\right)^2\right]^{\frac{3}{2}}} \right) du +$$

$$+ \left( \frac{18v\left(u^2 - v^2\right)}{\left[1 + 9\left(u^2 - v^2\right)^2\right]^{\frac{3}{2}}},\quad 0,\quad \frac{-6v}{\left[1 + 9\left(u^2 - v^2\right)^2\right]^{\frac{3}{2}}} \right) dv,$$

$$E_3 = \left( \frac{-3u^2 + 3v^2}{\sqrt{1 + 9\left(u^2 + v^2\right)^2}}, \frac{6uv}{\sqrt{1 + 9\left(u^2 + v^2\right)^2}}, \frac{1}{\sqrt{1 + 9\left(u^2 + v^2\right)^2}} \right),$$

$$\partial_u E_1 \cdot E_3 = \frac{18 \cdot 3u\left(u^2 - v^2\right)^2 + 6u}{\left[1 + 9\left(u^2 - v^2\right)^2\right]^{\frac{3}{2}} \sqrt{1 + 9\left(u^2 + v^2\right)^2}} du = \frac{6u\left[1 + 9\left(u^2 - v^2\right)^2\right]}{\left[1 + 9\left(u^2 - v^2\right)^2\right]^{\frac{3}{2}} \sqrt{1 + 9\left(u^2 + v^2\right)^2}} du,$$

$$\partial_v E_1 \cdot E_3 = \frac{-3\cdot 18v\left(u^2-v^2\right)^2-6v}{\left[1+9\left(u^2-v^2\right)^2\right]^{\frac{3}{2}}\sqrt{1+9\left(u^2+v^2\right)^2}}\,dv = \frac{-6v\left[1+9\left(u^2-v^2\right)^2\right]}{\left[1+9\left(u^2-v^2\right)^2\right]^{\frac{3}{2}}\sqrt{1+9\left(u^2+v^2\right)^2}}\,dv$$

$$= \frac{-6v\,dv}{\sqrt{1+9\left(u^2-v^2\right)^2}\,\sqrt{1+9\left(u^2+v^2\right)^2}}\,.$$

The result is

$$\omega_{13} = \frac{6u\,du-6v\,dv}{\sqrt{1+9\left(u^2-v^2\right)^2}\,\sqrt{1+9\left(u^2+v^2\right)^2}}\,.$$

The differential of the form $d\omega_{12}=\omega_{13}\wedge\omega_{32}$ is essential for calculation of the Gauss curvature. Let us study the following equations:

$$d\,\omega_{12} = \omega_{13}\wedge\omega_{32} =$$

$$= \frac{6u\,du-6v\,dv}{\sqrt{1+9\left(u^2-v^2\right)^2}\,\sqrt{1+9\left(u^2+v^2\right)^2}} \wedge \frac{6v\left[1-9(u^4-v^4)\right]du+6u\left[1+9\left(u^4-v^4\right)\right]dv}{\sqrt{1+9\left(u^2-v^2\right)^2}\,\left(1+9\left(u^2+v^2\right)^2\right)} =$$

$$= \frac{36u^2\left[1+9\left(u^4-v^4\right)\right]du\wedge dv+36v^2\left[1-9\left(u^4-v^4\right)\right]}{\left[1+9\left(u^2-v^2\right)^2\right]\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}du\wedge dv$$

$$= \frac{36\left[u^2+u^2\cdot 9\left(u^4-v^4\right)+v^2-v^2\cdot 9\left(u^4-v^4\right)\right]}{\left[1+9\left(u^2-v^2\right)^2\right]\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}du\wedge dv$$

$$= \frac{36\left[\left(u^2+v^2\right)+\left(u^2-v^2\right)\cdot 9\,(u^2+v^2)(u^2-v^2)\right]}{\left[1+9\left(u^2-v^2\right)^2\right]\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}du\wedge dv$$

$$= \frac{36\left(u^2+v^2\right)\left[1+9\left(u^2-v^2\right)^2\right]}{\left[1+9\left(u^2-v^2\right)^2\right]\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}\cdot\frac{1}{\sqrt{(1+9(u^2+v^2)}}\theta_1\wedge\theta_2$$

$$= \frac{36(u^2+v^2)}{\left[1+9\left(u^2+v^2\right)^2\right]^2} = -K,$$

where $K$ is the Gaussian curvature. We have

$$\omega_{31} \wedge \omega_{32} = K = \frac{-36(u^2+v^2)}{\left[1+9\left(u^2+v^2\right)^2\right]^2}$$

.

## 2 Weingarten method

Equation $E_i \cdot E_j = \delta_{ij}, \quad i, j = 1,2,3$ gives $dE_i \cdot E_j = 0$, which means $\partial_u E_3 \in T_x(M)$, $\partial_v E_3 \in T_x(M)$.

We have

$$\partial_u E_3 \cdot x_u = \frac{-6u}{\sqrt{1+9\left(u^2+v^2\right)^2}} \ ,$$

$$\partial_u E_3 \cdot x_v = \frac{6v}{\sqrt{1+9\left(u^2+v^2\right)^2}} \ ,$$

$$\partial_u E_3 = \beta_{11} \cdot x_u + \beta_{12} \cdot x_v \ ,$$

$$\partial_u E_3 \cdot x_u = \beta_{11} x_u \cdot x_u + \beta_{12} x_v \cdot x_u \ ,$$

$$\partial_u E_3 \cdot x_v = \beta_{11} x_u \cdot x_v + \beta_{12} x_v \cdot x_v \ ,$$

$$\frac{-6u}{\sqrt{1+9\left(u^2+v^2\right)^2}} = \beta_{11}\left(1+9\left(u^2-v^2\right)^2\right) + \beta_{12}\left(-18uv\left(u^2-v^2\right)\right),$$

$$\frac{6v}{\sqrt{1+9\left(u^2+v^2\right)^2}} = \beta_{11}\left(-18uv\left(u^2-v^2\right)^2\right) + \beta_{12}\left(1+36u^2v^2\right),$$

We have system of two equations for $\beta_{11}$, $\beta_{12}$. Thanks to Cramer's rule we obtain

$$\begin{vmatrix} 1+9\left(u^2-v^2\right)^2 & -18uv(u^2-v^2) \\ -18uv(u^2-v^2) & (1+36u^2v^2) \end{vmatrix} = 1+9\left(u^2+v^2\right)^2 \ .$$

Further we have

$$\beta_{11} = \begin{vmatrix} \dfrac{-6u}{\sqrt{1+9\left(u^2+v^2\right)^2}} & -18uv(u^2-v^2) \\ \dfrac{6v}{\sqrt{1+9\left(u^2+v^2\right)^2}} & 1+36u^2v^2 \end{vmatrix} = \frac{-6u\left[\left(1+36u^2v^2\right)+6v\cdot 18uv\left(u^2-v^2\right)\right]}{\left(1+9\left(u^2+v^2\right)^2\right)^{\frac{3}{2}}} =$$

$$= \frac{-6u\left[1+36u^2v^2-18u^2v^2+18v^4\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}} = \frac{-6u\left[1+18v^2(u^2+v^2)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}},$$

$$\beta_{12} = \begin{vmatrix} 1+9\left(u^2-v^2\right)^2 & \dfrac{-6u}{\sqrt{1+9\left(u^2+v^2\right)^2}} \\ -18uv(u^2-v^2) & \dfrac{6v}{\sqrt{1+9\left(u^2+v^2\right)^2}} \end{vmatrix} = \frac{6v\left[1+9\left(u^2-v^2\right)^2\right]-6u\cdot 18uv(u^2-v^2)}{\sqrt{1+9\left(u^2+v^2\right)^2}\;\left(1+9\left(u^2+v^2\right)^2\right)} =$$

$$= \frac{6v\left[1+9u^4-18u^2v^2+9v^4-18u^4+18u^2v^2\right]}{\sqrt{1+9\left(u^2+v^2\right)^2}\;\left(1+9\left(u^2+v^2\right)^2\right)} = \frac{6v\left[1-9\left(u^4-v^4\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}.$$

We have

$$\beta_{11} = \frac{-6u\left[1+18v^2\left(u^2+v^2\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}},$$

$$\beta_{12} = \frac{6v\left[1-9\left(u^4-v^4\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}.$$

Analogically we obtain

$$\partial_v E_3 \cdot x_u = \frac{6v}{\sqrt{1+9\left(u^2+v^2\right)^2}},$$

$$\partial_v E_3 \cdot x_v = \frac{6u}{\sqrt{1+9\left(u^2+v^2\right)^2}},$$

$$\partial_v E_3 = \beta_{21}x_u + \beta_{22}x_v,$$

$$\frac{6v}{\sqrt{1+9\left(u^2+v^2\right)^2}}=\beta_{21}\left[1+9\left(u^2-v^2\right)^2\right]+\beta_{22}\left[-18uv\left(u^2-v^2\right)\right],$$

$$\partial_v E_3\cdot x_u=\frac{6u}{\sqrt{1+9\left(u^2+v^2\right)^2}},$$

$$\partial_v E_3\cdot x_v=\beta_{21}x_u\cdot x_v+\beta_{22}x_v\cdot x_v,$$

$$\frac{6u}{\sqrt{1+9\left(u^2+v^2\right)^2}}=\beta_{21}\left[-18uv(u^2-v^2)\right]+\beta_{22}\left[1+36u^2v^2\right],$$

$$D=\begin{vmatrix}1+9\left(u^2-v^2\right)^2 & -18uv(u^2-v^2)\\ -18uv(u^2-v^2) & 1+36u^2v^2\end{vmatrix}=1+9\left(u^2+v^2\right)^2$$

$$\beta_{21}=\begin{vmatrix}\dfrac{6v}{\sqrt{1+9\left(u^2+v^2\right)^2}} & -18uv(u^2-v^2)\\ \dfrac{6u}{\sqrt{1+9\left(u^2+v^2\right)^2}} & 1+36u^2v^2\end{vmatrix}\cdot\frac{1}{1+9\left(u^2+v^2\right)^2}=$$

$$=\frac{6v\left[1+36u^2v^2+18u^2(u^2-v^2)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}=\frac{6v\left[1+36u^2v^2-18u^2v^2+18u^4\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}=$$

$$=\frac{6v\left[1+18u^2(u^2+v^2)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}.$$

$$\beta_{22}=\begin{vmatrix}1+9\left(u^2-v^2\right)^2 & \dfrac{6v}{\sqrt{1+9\left(u^2+v^2\right)^2}}\\ -18uv\left(u^2-v^2\right) & \dfrac{6u}{\sqrt{1+9\left(u^2+v^2\right)^2}}\end{vmatrix}\cdot\frac{1}{1+9\left(u^2+v^2\right)^2}=$$

$$= \frac{6u\left[1+9u^4-18u^2v^2+9v^4+18v^2(u^2-v^2)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}} = \frac{6u\left[1+9u^4+9v^4-18v^4\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}} =$$

$$= \frac{6u\left[1+9\left(u^4-v^4\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}},$$

Weingarten map can be represented by the matrix $\mathcal{W}$

$$\mathcal{W} = \begin{vmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{vmatrix} = \begin{vmatrix} \frac{-6u\left[1+18v^2\left(u^2+v^2\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}} & \frac{6v\left[1-9\left(u^4-v^4\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}} \\ \frac{6v\left[1+18u^2(u^2+v^2)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}} & \frac{6u\left[1+9\left(u^4-v^4\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}} \end{vmatrix}.$$

As $W(x_u) = -\partial_u E_3$ and $W(x_v) = -\partial_v E$ we obtain:

$$-\mathcal{W} = \begin{vmatrix} \frac{6u\left[1+18v^2\left(u^2+v^2\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}} & \frac{-6v\left[1-9\left(u^4-v^4\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}} \\ \frac{-6v\left[1+18u^2\left(u^2+v^2\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}} & \frac{-6u\left[1+9\left(u^4-v^4\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}} \end{vmatrix}.$$

$$det(-\mathcal{W}) = \frac{-36u^2\left[1+18v^2\left(u^2+v^2\right)\right]\left[1+9\left(u^4-v^4\right)\right]-36v^2\left[1+18u^2\left(u^2+v^2\right)\right]\left[1-9\left(u^4-v^4\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^3}$$

$$= \frac{-36\left[u^2+18u^2v^2\left(u^2+v^2\right)+9u^2\left(u^4-v^4\right)+18\cdot 9u^2v^2\left(u^4-v^4\right)+v^2\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^3}+$$

$$+\frac{\left[18u^2v^2\left(u^2+v^2\right)-9v^2\left(u^4-v^4\right)-18\cdot 9\,u^2v^2(u^4-v^4)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^3}$$

$$= \frac{-36\left[\left(u^2+v^2\right)+36u^2v^2\left(u^2+v^2\right)+9\left(u^2-v^2\right)^2\left(u^2+v^2\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^3}=$$

$$= \frac{-36\left(u^2+v^2\right)\left[1+36u^2v^2+9u^4-18u^2v^2+9v^4\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^3}$$

$$= \frac{-36\left(u^2+v^2\right)\left[1+9\left(u^2+v^2\right)^2\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^3} = \frac{-36\left(u^2+v^2\right)}{\left[1+9\left(u^2+v^2\right)^2\right]^2}$$

As was given before, the Gaussian curvature is $K = \dfrac{-36\left(u^2+v^2\right)}{\left[1+9\left(u^2+v^2\right)^2\right]^2}$.

Further we obtain

$$H=\frac{1}{2}\ tr(-\mathcal{W})=\frac{1}{2}\frac{6u\left[1+18v^2\left(u^2+v^2\right)-1-9\left(u^4-v^4\right)\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}=\frac{27u\left[2u^2v^2+2v^4-u^4+v^4\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}=$$

$$=\frac{27u\left[2u^2v^2+3v^4-u^4\right]}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}=\frac{54u^3v^2+81uv^4-27u^5}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}.$$

So the trace of the matrix $(-\mathcal{W})$ is

$$H=\frac{1}{2}tr\left(-\mathcal{W}\right)=\frac{54u^3v^2+81uv^4-27u^5}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}.$$

## Conclusion

By the method of moving frame we reached that the result of Gaussian curvature is

$$K=\frac{-36(u^2+v^2)}{\left[1+9\left(u^2+v^2\right)^2\right]^2}.$$

By the method of Weingarten mapping we reached that the result of Main curvature is

$$H=\frac{1}{2}tr\left(-\mathcal{W}\right)=\frac{54u^3v^2+81uv^4-27u^5}{\left[1+9\left(u^2+v^2\right)^2\right]^{\frac{3}{2}}}.$$

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