APPLICATION OF CARTAN´S MOVING FRAME METHOD

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Abstract

The aim of this article is to give basic geometrical characteristic of sphere and torus, but mainly of Cobb-Douglas functions used in economics. We are going to study these functions as regular surfaces in $\mathbb{R}^3$. Applying the method of Cartan moving frame we obtain geometrical description of Cobb-Douglas function used in economy

$$x(u,v) = A \cdot u^{\alpha} \cdot v^{\beta}, \quad \text{where } A = 1,\ u > 0,\ v > 0 \text{ and } \alpha, \beta \in \mathbb{R}.$$

Key words: Orthonormal frame, tangent space, Gaussian curvature, Mean curvature, Maurer-Cartan equations, Cartan's lemma

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Introduction

Let $U \subset \mathbb{R}^2$ be an open neighbourhood of a point $(u,v) \in U$ and $x: U \to \mathbb{R}^3$ a regular map. A subset $M \subset \mathbb{R}^3$ is called a regular two dimensional surface in $\mathbb{R}^3$ if for each $p \in M$ there exist an open neighbourhood V of $p \in \mathbb{R}^3$ and a map $x: U \to V \cap M$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap M$ such that x is a differentiable homeomorphism and the differential $dx_q : T_q(U) \to T_{x(q)}(M)$ is injective for all $q \in U$, $x(q) = p$. Then it is possible to choose in $x(U)$ an orthonormal moving frame $\{E_1, E_2, E_3\}$ in such a way that E_1, E_2 are tangent to $x(U)$ and E_3 is a non-vanishing normal to $x(U)$.

1 Basic equations

We first discuss the Cartan structural equations for a two-dimensional surface in $\mathbb{R}^3$.
Differentiating a map $x(u,v)$ we obtain

$$dx = x_u\, du + x_v\, dv,$$

where x_u, x_v are tangent vector fields. Let us denote moving frame $\{x_u, x_v, n\}$ where n is a normal vector field, and

$$N(u,v) = \frac{x_u \times x_v}{\|x_u \times x_v\|}$$

is a unit normal field. With respect to the orthonormal moving frame $\{E_1, E_2, E_3\}$ we define forms

$$\theta_i = E_i\, dx = E_i\, x_u\, du + E_i\, x_v\, dv, \quad i = 1, 2, 3 \tag{1}$$

Since x_u and x_v are tangent to $x(U)$ we have $E_3\, dx = N\, dx = 0$ which implies $\theta_3 = 0$.

Each vector $E_i : U \subset \mathrm{R}^3 \to \mathrm{R}^3$ is a differentiable function and the differential

$$dE_i \,:\, T_q(U) \to T_{x(q)}(M)$$

is a linear map. We may write (using Einstein's notation)

$$dE_i = \omega_{ij} E_j\,,$$

where ω_{ij} are linear forms on R^3 and since E_i are differentiable ω_{ij} are nine differentiable forms. We have

$$\begin{pmatrix} dE_1 \\ dE_2 \\ dE_3 \end{pmatrix} = \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix} \cdot \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}. \tag{2}$$

Differentiating equation $E_i \cdot E_j = \delta_{ij}$ where δ_{ij} is the Kronecker's symbol, we obtain

$$dE_i E_j + E_i dE_j = \omega_{ij} + \omega_{ji} = 0.$$

Forms ω_{ij} are antisymmetric

$$\omega_{ii} = 0,\ \omega_{ij} = -\omega_{ji}\ . \tag{3}$$

From (2) and (3) follows

$$\begin{pmatrix} dE_1 \\ dE_2 \\ dE_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix} \cdot \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}. \tag{4}$$

Forms dx and dE_i have vanishing exterior derivatives, which means

$$d^2x = 0 \quad \text{and} \quad d^2E_i = 0, \text{ where } i = 1,2,3.$$

So we have

$$0 = d^2x = dE_1 \wedge \theta_1 + E_1 d\theta_1 + dE_2 \wedge \theta_2 + E_2 d\theta_2. \tag{5}$$

Substituting (4) into (5) we obtain

$$(\omega_{12}E_2 + \omega_{13}E_3) \wedge \theta_1 + E_1 d\theta_1 + (\omega_{21}E_1 + \omega_{23}E_3) \wedge \theta_2 + E_2 d\theta_2 = 0. \tag{6}$$

From (6) follows

$$(d\theta_1 + \omega_{21} \wedge \theta_2) E_1 + (d\theta_2 + \omega_{12} \wedge \theta_1) E_2 + (\omega_{13} \wedge \theta_1 + \omega_{23} \wedge \theta_2) E_3 = 0. \tag{7}$$

The linear independence of vectors E_1, E_2, E_3 and equation (7) gives the following equations:

$$d\theta_1 = \omega_{12} \wedge \theta_2, \tag{8}$$
$$d\theta_2 = \omega_{21} \wedge \theta_1, \tag{9}$$
$$0 = \omega_{13} \wedge \theta_1 + \omega_{23} \wedge \theta_2. \tag{10}$$

Exterior derivatives (4) gives:

$$0 = d^2 E_1 = d\omega_{12}E_2 - \omega_{12} \wedge dE_2 + d\omega_{13}E_3 - \omega_{13} \wedge dE_3,$$
$$d\omega_{12}E_2 - \omega_{12} \wedge (\omega_{21}E_1 + \omega_{23}E_3) + \omega_{13}E_3 - \omega_{13} \wedge (\omega_{31}E_1 + \omega_{32}E_2) = 0,$$

we have

$$(d\omega_{12} - \omega_{13} \wedge \omega_{32})E_2 + (d\omega_{13} - \omega_{12} \wedge \omega_{23})E_3 = 0. \tag{11}$$

From (11) follows

$$\begin{aligned} d\omega_{12} &= \omega_{13} \wedge \omega_{32}, \\ d\omega_{13} &= \omega_{12} \wedge \omega_{23}. \end{aligned} \tag{12}$$

Analogically:

$$\begin{aligned} d^2 E_2 = d\omega_{21}E_1 - \omega_{21} \wedge dE_1 + d\,\omega_{23}E_3 - \omega_{23} \wedge dE_3 = 0, \\ d\omega_{21}E_1 - \omega_{21} \wedge (\omega_{12}E_2 + \omega_{13}E_3) + \omega_{23}E_3 - \omega_{23} \wedge (\omega_{31}E_1 + \omega_{32}E_2) = 0, \\ (d\,\omega_{23} - \omega_{21} \wedge \omega_{13}) E_3 + (d\,\omega_{21} - \omega_{23} \wedge \omega_{31}) E_1 = 0. \end{aligned} \tag{13}$$

From (13) follows

$$\begin{aligned} d\omega_{23} &= \omega_{21} \wedge \omega_{13}, \\ d\omega_{21} &= \omega_{23} \wedge \omega_{31}. \end{aligned} \tag{14}$$

Equations (8), (9), (10), (12) and (14) are called Maurer-Cartan structural equations. From equation (10) and Cartan's lemma follow

s

$$\omega_{13} = \alpha_{11}\theta_1 + \alpha_{12}\theta_2, \quad \omega_{23} = \alpha_{12}\theta_1 + \alpha_{22}\theta_2. \tag{15}$$

From (15) and (12) we have

$$\begin{aligned} d\,\omega_{12} = \omega_{13} \wedge \omega_{32} = -\omega_{13} \wedge \omega_{23} = \\ = -(\alpha_{11}\theta_1 + \alpha_{12}\theta_2) \wedge (\alpha_{12}\theta_1 + \alpha_{22}\theta_2). \end{aligned} \tag{16}$$

Equation (16) gives

$$d\omega_{12} = -\left(\alpha_{11}\alpha_{22} - \alpha_{12}^2 \right) \theta_1 \wedge \theta_2 = -K\,\theta_1 \wedge \theta_2,$$

where $K = \alpha_{11}\alpha_{22} - \alpha_{12}^2$ is the Gaussian curvature.

2.Examples

2.1 Example 1. Sphere $S^2 \subset \mathbb{R}^3$

Local parametrization of the sphere $S^2 \subset \mathbb{R}^3$ is given by the map

$$x(u, v) = (r\cos v\cos u, r\cos v\sin u, r\sin v), \quad \text{where} \quad (u, v) \in (0, 2\pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

The moving frame is

$$\begin{aligned} x_u &= (-r\cos v\sin u, r\cos v\cos u, 0) = r\cos v(-\sin u, \cos u, 0), \\ x_v &= r(-\sin v\cos u, -\sin v\sin u, \cos v), \\ n &= r^2 \cdot \cos v(\cos u\cos v, \sin u\cos v, \sin v). \end{aligned}$$

Orthonormal moving frame is

$$\begin{aligned} E_1 &= (-\sin u, \cos u, 0), \\ E_2 &= (-\sin v\cos u, -\sin v\sin u, \cos v), \\ E_3 &= (\cos u\cos v, \sin u\cos v, \sin v). \end{aligned}$$

The differential dE_1 equals

$$\begin{aligned} \partial_u E_1 &= (-\cos u, -\sin u, 0), \\ \partial_v E_1 &= (0,0,0), \\ dE_1 &= (-\cos u, -\sin u, 0)du. \end{aligned}$$

Forms ω_{12} and ω_{13} are

$$\begin{aligned} \omega_{12} &= dE_1 \cdot E_2 = (\sin v\cos^2 u + \sin v\sin^2 u)du = \sin v\, du, \\ \omega_{13} &= dE_1 \cdot E_3 = (-\cos v\cos^2 u - \cos v\sin^2 u)du = -\cos v\, du. \end{aligned}$$

Analogically we have

$$\begin{aligned} \partial_u E_2 &= (\sin v\sin u, -\sin v\cos u, 0), \\ \partial_v E_2 &= (-\cos v\cos u, -\cos v\sin u, -\sin v). \\ dE_2 &= (\sin v\sin u, -\sin v\cos u, 0)du + (-\cos v\cos u, -\cos v\sin u, -\sin v)dv. \end{aligned}$$

Forms ω_{21} and ω_{23} are

$$\begin{aligned} \omega_{21} &= dE_2 \cdot E_1 = \left(-\sin v\sin^2 u - \sin v\cos^2 u\right)du + (\sin u\cos v\cos u - \cos u\cos v\sin u)\,dv \\ &= -\sin v\, du, \\ \omega_{23} &= dE_2 \cdot E_3 = (\sin v\sin u\cos u\cos v - \sin v\cos u\sin u\cos v)\,du \\ &\quad + \left(-\cos v\cos u \cdot \cos u\cos v - \cos v\sin u\sin u\cos v - \sin^2 v\right)dv \\ &= \left(-\cos^2 v\cos^2 u - \cos^2 v\sin^2 u - \sin^2 v\right)dv = \\ &= \left(-\cos^2 v(\cos^2 u + \sin^2 u) - \sin^2 v\right)dv = -dv. \end{aligned}$$

The differential dE_3 equals

$$\partial_u E_3 = (-\sin u \cos v, \cos u \cos v, 0),$$
$$\partial_v E_3 = (-\cos u \sin v, -\sin u \sin v, \cos v),$$

$$dE_3 = (-\sin u \cos v, \cos u \cos v, 0)du + (-\cos u \sin v, -\sin u \sin v, \cos v)dv.$$

Forms ω_{31} and ω_{32} are

$$\begin{aligned}
\omega_{31} &= dE_3 \cdot E_1 = \\
&= \left(\sin^2 u \cos v + \cos^2 u \cos v\right)du + \\
&\quad + \left(\sin u \cos u \sin v - \sin u \sin v \cos u\right) dv \\
&= \cos v \, du, \\
\omega_{32} &= dE_3 \cdot E_2 \\
&= \left(\sin u \cos v \sin v \cos u - \cos u \cos v \sin v \sin u\right) du + \\
&\quad + (\cos^2 u \sin^2 v + \sin^2 u \sin^2 v + \cos^2 v)\, dv \\
&= dv.
\end{aligned}$$

The forms θ_1 and θ_2 are:

$$\begin{aligned}
\theta_1 &= E_1\left(x_u du + x_v dv\right) = \\
&= (-\sin u, \cos u, 0)(-r \cos v \sin u, r \cos v \cos u, 0)\, du + \\
&\quad + (-\sin u, \cos u, 0)(-r \sin v \cos u, -r \sin v \sin u, r \cos v)\, dv \\
&= (r \sin^2 u \cos v + r \cos^2 u \cos v)\, du + \\
&\quad + \left(r \sin u \cos u \sin v - r \sin v \sin u \cos u\right) dv \\
&= r \cos v \, du, \\
\theta_2 &= E_2\left(x_u du + x_v dv\right) = \\
&= (-\sin v \cos u, -\sin v \sin u, \cos v)(-r \sin u \cos v, r \cos u \cos v, 0)\, du + \\
&\quad + (-\sin v \cos u, -\sin v \sin u, \cos v)(-r \cos u \sin v, -r \sin u \sin v, r \cos v)\, dv \\
&= \left(r \sin v \cos u \sin u \cos v - r \sin v \sin u \cos u \cos v\right) du + \\
&\quad + \left(r \sin^2 v \cos^2 u + r \sin^2 v \sin^2 u + r \cos^2 v\right) dv \\
&= r \, dv.
\end{aligned}$$

From exterior product $\theta_1 \wedge \theta_2 = r^2 \cos v \, du \wedge dv$ follows $du \wedge dv = \dfrac{1}{r^2 \cos v} \theta_1 \wedge \theta_2$.

Further we have

$$\omega_{31} \wedge \omega_{32} = \cos v \, du \wedge dv,$$

and

$$\omega_{31} \wedge \omega_{32} = \cos v \frac{1}{r^2 \cos v} \theta_1 \wedge \theta_2 = \frac{1}{r^2} \theta_1 \wedge \theta_2.$$

The equation

$$\omega_{31} \wedge \omega_{32} = K \cdot \theta_1 \wedge \theta_2,$$

where K is Gaussian curvature gives

$$K = \frac{1}{r^2}.$$

2.2 Example 2. Torus $T^2 \subset \mathrm{R}^3$

Local parametrization of the torus in R^3 is given by the map

$$x(u,v) = \left((a + b\cos v)\cos u, (a + b\cos v)\sin u, b\sin v \right).$$

The moving frame has the form

$$\begin{aligned} x_u &= (a + b\cos v)(-\sin u, \cos u, 0), \\ x_v &= b(-\sin v\cos u, -\sin v\sin u, \cos v), \\ n &= (\cos u\cos v, \sin u\cos v, \sin v). \end{aligned}$$

The orthonormal frame can be written in the form

$$\begin{aligned} E_1 &= (-\sin u, \cos u, 0), \\ E_2 &= (-\sin v\cos u, -\sin v\sin u, \cos v), \\ E_3 &= (\cos u\cos v, \sin u\cos v, \sin v). \end{aligned}$$

$$\begin{aligned} \partial_u E_1 &= (-\cos u, -\sin u, 0) \\ \partial_v E_1 &= (0,0,0). \end{aligned}$$

So $dE_1 = (-\cos u, -\sin u, 0)\, du.$

Forms ω_{12} and ω_{13} are

$$\begin{aligned} \omega_{12} &= dE_1 \cdot E_2 = \\ &= (-\cos u, -\sin u, 0)\, du \cdot (-\sin v\cos u, -\sin v\sin u, \cos v) = \\ &= \left(\sin v\cos^2 u + \sin v\sin^2 u + 0\right) du = \sin v\, du, \\ \omega_{13} &= dE_1 \cdot E_3 = \\ &= (-\cos u, -\sin u, 0) \cdot (\cos u\cos v, \sin u\cos v, \sin v)\, du = \\ &= -\cos^2 u\cos v - \sin^2 u\cos v = -\cos v\, du. \end{aligned}$$

Further we have

$$\begin{aligned}\theta_1 &= E_1(x_u du + x_v dv) = \\ &= (-\sin u, \cos u, 0)\cdot[(a+b\cos v)(-\sin u, \cos u, 0)du + b(-\sin v\cos u, -\sin v\sin u, \cos v)dv] = \\ &= (a+b\cos v) + b[\sin u\cos u\sin v - \sin u\cos u\sin v + 0] = \\ &= (a+b\cos v)du\end{aligned}$$

$$\begin{aligned}\theta_2 &= E_2(x_u du + x_v dv) = \\ &= (-\sin v\cos u, -\sin v\sin u, \cos v)\cdot \\ &\cdot[(a+b\cos v)(-\sin u, \cos u, 0)du + b(-\sin v\cos u, -\sin v\sin u, \cos v)dv] \\ &= (a+b\cos v)[\sin v\cos u\sin u - \sin v\sin u\cos u + 0]du + \\ &+ b\left[\sin^2 v\cos^2 u + \sin^2 v\sin^2 u + \cos^2 v\right]dv \\ &= b\left[\sin^2 v\left(\cos^2 u + \sin^2 u\right) + \cos^2 v\right]dv = b\, dv.\end{aligned}$$

We have

$$\begin{aligned}\omega_{12} &= \sin v\, du, \\ \theta_1 &= (a+b\cos v)du, \quad \theta_2 = b\ dv, \\ \theta_1\wedge\theta_2 &= b(a+b\cos v)du\wedge dv, \\ du\wedge dv &= \frac{1}{b(a+b\cos v)}\theta_1\wedge\theta_2, \\ d\omega_{12} &= -\cos v\ du\wedge dv, \\ d\omega_{12} &= -\frac{\cos v}{b(a+b\cos v)}\ \theta_1\wedge\theta_2.\end{aligned}$$

The equation

$$d\omega_{12} = -K\cdot\ \theta_1\wedge\theta_2,$$

gives the formula for Gaussian curvature K.

$$K = \frac{\cos v}{b(a+b\cos v)}.$$

2.3 Example 3. Gaussian curvature of Cobb-Douglas surfaces

Let $x(u,v) = (u, v, u^\alpha\cdot v^\beta)$, where $u > 0$, $v > 0$ and $\alpha,\beta\in\mathrm{R}$ are studied Cobb Douglas surfaces:

$$\begin{aligned}x(u,v) &= \left(u, v, u^\alpha\cdot v^\beta\right) \\ \left(u^\alpha\cdot v^\beta\right)_u &= \alpha\cdot u^{\alpha-1}\cdot v^\beta, \\ \left(u^\alpha\cdot v^\beta\right)_v &= \beta\cdot u^\alpha\cdot v^{\beta-1}, \\ \left(u^\alpha\cdot v^\beta\right)_{uu} &= \alpha\cdot(\alpha-1)\cdot u^{\alpha-2}v^\beta, \\ \left(u^\alpha\cdot v^\beta\right)_{vv} &= \beta\cdot(\beta-1)\cdot u^\alpha v^{\beta-2}, \\ \left(u^\alpha\cdot v^\beta\right)_{uv} &= \alpha\cdot\beta\cdot u^{\alpha-1}\cdot v^{\beta-1}.\end{aligned}$$

The moving frame is

$$x_u = \left(1,\ 0,\ \alpha \cdot u^{\alpha-1} \cdot v^{\beta}\right),$$
$$x_v = \left(0,\ 1,\ \beta \cdot u^{\alpha} \cdot v^{\beta-1}\right),$$
$$n = \left(-\alpha \cdot u^{\alpha-1} \cdot v^{\beta},\ -\beta \cdot u^{\alpha} \cdot v^{\beta-1},\ 1\right).$$

Let us denote

$$A = 1 + \alpha^2 \cdot u^{2\alpha-2} \cdot v^{2\beta},$$
$$B = 1 + \alpha^2 \cdot u^{2\alpha-2} \cdot v^{2\beta} + \beta^2 \cdot u^{2\alpha} \cdot v^{2\beta-2}.$$

Orthonormal moving frame has the form

$$E_1 = \left(\frac{1}{\sqrt{A}},\quad 0,\quad \frac{\alpha \cdot u^{\alpha-1} \cdot v^{\beta}}{\sqrt{A}}\right),$$
$$E_2 = \left(\frac{-\alpha\beta \cdot u^{2\alpha-1} \cdot v^{2\beta-1}}{\sqrt{A}\sqrt{B}},\ \frac{A}{\sqrt{A}\sqrt{B}},\ \frac{\beta \cdot u^{\alpha} \cdot v^{\beta-1}}{\sqrt{A}\sqrt{B}}\right),$$
$$E_3 = \left(\frac{-\alpha \cdot u^{\alpha-1} \cdot v^{\beta}}{\sqrt{B}},\ \frac{-\beta \cdot u^{\alpha} \cdot v^{\beta-1}}{\sqrt{B}},\ \frac{1}{\sqrt{B}}\right).$$

We have

$$\partial_u E_1 = \left(\frac{-\alpha^2 \cdot (\alpha-1) \cdot u^{2\alpha-3} \cdot v^{2\beta}}{A^{3/2}},\ 0,\ \frac{\alpha \cdot (\alpha-1) \cdot u^{\alpha-2} \cdot v^{\beta}}{A^{3/2}}\right),$$
$$\partial_v E_1 = \left(\frac{-\alpha^2 \cdot \beta \cdot u^{2\alpha-2} \cdot v^{2\beta-1}}{A^{3/2}},\ 0,\ \frac{\alpha \cdot \beta \cdot u^{\alpha-1} \cdot v^{\beta-1}}{A^{3/2}}\right),$$
$$dE_1 = \partial_u E_1\, du + \partial_v E_1\, dv,$$
$$\omega_{12} = dE_1 \cdot E_2 = \frac{\left[\alpha \cdot (\alpha-1) \cdot \beta \cdot u^{2\alpha-2} \cdot v^{2\beta-1}\right] du + \left[\alpha \cdot \beta^2 \cdot u^{2\alpha-1} \cdot v^{2\beta-2}\right] dv}{A\sqrt{B}},$$
$$\omega_{31} = dE_3 \cdot E_1 = \frac{-\left[\alpha \cdot (\alpha-1) \cdot u^{\alpha-2} \cdot v^{\beta}\right] du - \left[\alpha \cdot \beta^2 \cdot u^{\alpha-1} \cdot v^{\beta-1}\right] dv}{\sqrt{A} \cdot \sqrt{B}}.$$

Analogically we obtain

$$\omega_{32} = \frac{\alpha^2 \cdot (\alpha-1) \cdot \beta \cdot u^{3\alpha-3} \cdot v^{3\beta-1} - \alpha \cdot \beta \cdot u^{\alpha-1} \cdot v^{\beta-1} \cdot A}{\sqrt{A}\,B}\ du +$$
$$+ \frac{\alpha^2 \cdot \beta^2 \cdot u^{3\alpha-2} \cdot v^{3\beta-2} - \beta \cdot (\beta-1) \cdot u^{\alpha} \cdot v^{\beta-2} \cdot A}{\sqrt{A}\,B}\ dv.$$

The exterior product of ω_{31} and ω_{32} is

$$\omega_{31} \wedge \omega_{32} = \frac{-\alpha^3 \cdot (\alpha-1) \cdot \beta^2 \cdot u^{4\alpha-4} \cdot v^{4\beta-2} + \alpha \cdot \beta \cdot (\alpha-1) \cdot (\beta-1) \cdot u^{2\alpha-2} \cdot v^{2\beta-2} \cdot A}{A \cdot B^{3/2}} du \wedge dv +$$

$$+ \frac{\alpha^3 \cdot (\alpha-1) \cdot \beta^2 \cdot u^{4\alpha-4} \cdot v^{4\beta-2} - \alpha^2 \cdot \beta^2 \cdot u^{2\alpha-2} \cdot v^{2\beta-2} \cdot A}{A \cdot B^{3/2}} du \wedge dv$$

$$= \frac{\alpha \cdot \beta \cdot (\alpha-1) \cdot (\beta-1) \cdot u^{2\alpha-2} \cdot v^{2\beta-2} - \alpha^2 \cdot \beta^2 \cdot u^{2\alpha-2} \cdot v^{2\beta-2}}{B^{3/2}} du \wedge dv =$$

$$= \frac{\alpha \cdot \beta \cdot (\alpha \cdot \beta - \alpha - \beta + 1 - \alpha \cdot \beta) u^{2\alpha-2} \cdot v^{2\beta-2}}{B^{3/2}} du \wedge dv =$$

$$= \frac{\alpha \cdot \beta \cdot (-\alpha - \beta + 1) u^{2\alpha-2} \cdot v^{2\beta-2}}{B^{3/2}} du \wedge dv.$$

Further we have

$$\theta_i = E_i x_u du + E_i x_v dv, \quad i = 1,2.$$

Specially

$$\theta_1 = E_1 x_u du + E_1 x_v dv$$

$$= \left(\frac{1}{\sqrt{A}}, 0, \frac{\alpha \cdot u^{\alpha-1} \cdot v^{\beta}}{\sqrt{A}} \right) \left(1, 0, \alpha \cdot u^{\alpha-1} \cdot v^{\beta} \right) du + \left(\frac{1}{\sqrt{A}}, 0, \frac{\alpha \cdot u^{\alpha-1} \cdot v^{\beta}}{\sqrt{A}} \right) \left(0, 1, \beta \cdot u^{\alpha} \cdot v^{\beta-1} \right) dv =$$

$$= \sqrt{A}\, du + \frac{\alpha \cdot \beta \cdot u^{2\alpha-1} \cdot v^{2\beta-1}}{\sqrt{A}} dv.$$

Analogically we obtain

$$\theta_2 = E_2 x_u du + E_2 x_v dv =$$

$$= \left(-\frac{\alpha \cdot \beta \cdot u^{2\alpha-1} \cdot v^{2\beta-1}}{\sqrt{A}\sqrt{B}}, \frac{1+\alpha^2 \cdot u^{2\alpha-1} \cdot v^{2\beta}}{\sqrt{A}\sqrt{B}}, \frac{\beta \cdot u^{\alpha} \cdot v^{\beta-1}}{\sqrt{A}\sqrt{B}} \right) \left(1, 0, \alpha \cdot u^{\alpha-1} \cdot v^{\beta} \right) du +$$

$$+ \left(-\frac{\alpha \cdot \beta \cdot u^{2\alpha-1} \cdot v^{2\beta-1}}{\sqrt{A}\sqrt{B}}, \frac{1+\alpha^2 \cdot u^{2\alpha-1} \cdot v^{2\beta}}{\sqrt{A}\sqrt{B}}, \frac{\beta \cdot u^{\alpha} \cdot v^{\beta-1}}{\sqrt{A}\sqrt{B}} \right) \left(0, 1, \beta \cdot u^{\alpha} \cdot v^{\beta-1} \right) dv$$

$$= \frac{1+\alpha^2 \cdot u^{2\alpha-1} \cdot v^{2\beta} + \beta^2 \cdot u^{2\alpha} \cdot v^{2\beta-2}}{\sqrt{A}\sqrt{B}} dv.$$

Finally we have

$$\theta_1 \wedge \theta_2 = \sqrt{A} \frac{\sqrt{B}}{\sqrt{A}} du \wedge dv,$$

which means

$$du \wedge dv = \frac{1}{\sqrt{B}}\theta_1 \wedge \theta_2.$$

The final result is

$$\omega_{31} \wedge \omega_{32} = \frac{\alpha \cdot \beta \cdot (1-\alpha-\beta) \cdot u^{2\alpha-2} \cdot v^{2\beta-2}}{B^2}\theta_1 \wedge \theta_2.$$

The Gaussian curvature equals

$$K = \frac{\alpha \cdot \beta \cdot (1-\alpha-\beta) \cdot u^{2\alpha-2} \cdot v^{2\beta-2}}{(1+\alpha^2 \cdot u^{2\alpha-2} \cdot v^{2\beta} + \beta^2 \cdot u^{2\alpha} \cdot v^{2\beta-2})^2}\ .$$

Conclusion

Two economical examples served as an illustration of Maurer-Cartan equations and we reached the following results:

The Gaussian and Mean curvatures of the surface are :

Example 1:

$$K = \frac{1}{r^2}.$$

Example 2:

$$K = \frac{\cos v}{b \cdot (a + b\cos v)}.$$

Example 3:

$$K = \frac{\alpha \cdot \beta \cdot (1-\alpha-\beta) \cdot u^{2\alpha-2} \cdot v^{2\beta-2}}{(1+\alpha^2 \cdot u^{2\alpha-2} \cdot v^{2\beta} + \beta^2 \cdot u^{2\alpha} \cdot v^{2\beta-2})^2}.$$

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