REGULAR PARAMETRIC SURFACES IN R^3

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Abstract

The principal objects of this paper are regular parametrical surfaces in R^3. The method we are going to use is based on Weingarten mapping. We are going to suppose that the mapping $x: U \to \mathrm{R}^3$, where $(u,v) \in U \subset \mathrm{R}^2$ and $x(u,v) \in \mathrm{R}^3$, is regular. Symbols x_u and x_v are used in this paper instead of $\partial_u x$, $\partial_v x$ etc. These vectors form the basis of tangent space $T_{x(u,v)}M$ (see Fig.1). On $T_x(M)$ we can construct moving frame ($x_u,\ x_v,\ x_u \times x_v$). Vectors x_u and x_v are tangent vector fields of $T_x(M)$, $x_u \times x_v = n$ is a normal vector field and $N = \dfrac{x_u \times x_v}{\|x_u \times x_v\|}$ is a unit normal vector field.

Key words: Weingarten Map, First and Second fundamental forms, structural equations, Gaussian and Mean curvature

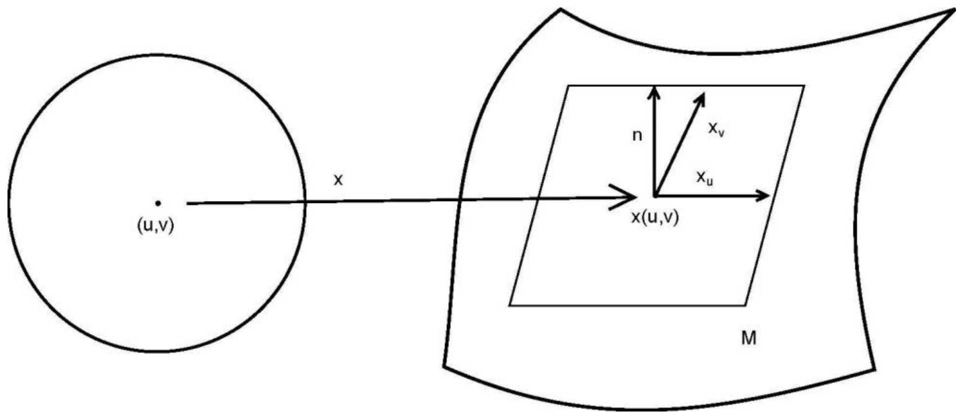
JEL Code: C00

Introduction

Let $U \subset \mathrm{R}^2$ is an open neighborhood of a point $(u, v) \in U$ and $x: U \to \mathrm{R}^3$ is a regular map (which means that the rank of Jacobian matrix $J(x)(u,v) = 2$). A subset $M \subset \mathrm{R}^3$ is called a regular two dimensional surface in R^3 if for each point $x = x(u,v)$ there exist an open neighborhood V of $x(u,v) \in \mathrm{R}^3$ and the map $x: U \subset \mathrm{R}^2 \to M \cap V$ of an open subset $U \subset \mathrm{R}^2$ onto $M \cap V$ is such that

1. x is a differentiable homeomorphism ,
2. the differential $dx_q: T_q(U) \to T_{x(q)}(M)$ is injective for all $q \in U$.

Fig. 1



1 Structural equations

Let U is an open neighborhood of the point $(u,v) \in \mathrm{R}^2$ and $x{:}U \to \mathrm{R}^3$ a regular map. Let $x(U) = M \cap V \subset \mathrm{R}^3$, where V is a neighborhood of the point $x(u,v)$.

$T_x(M)$ is the tangent plane to the surface M at the point $x = x(u,v)$. Tangent vectors x_u and x_v generate vector space $T_x(M)$ (see Fig. 1).

Let $N = \dfrac{x_u \times x_v}{\|x_u \times x_v\|}$ be a unit normal of the surface M. So we have

$$N \cdot x_u = 0, \qquad N \cdot x_v = 0, \qquad N \cdot N = 1. \tag{1}$$

From (1) follows that $N_u \in T_x(M)$ and $N_v \in T_v(M)$.

Remark 1. The first fundamental form of the surface is:

$$F_1(w_1, w_2) = w_1 \cdot w_2, \text{ where vectors } w_1, w_2 \in T_x(M).$$

In case $w_1 = w_2 = w$, we have

$$F_1(w,w) = w \cdot w, \quad \text{where vector } w \in T_x(M),$$
$$F_1(w,w) = (ax_u + bx_v) \cdot (ax_u + bx_v) = a^2 g_{11} + 2ab g_{12} + b^2 g_{22},$$

where $g_{11} = x_u \cdot x_u, \quad g_{12} = x_u \cdot x_v, \quad g_{22} = x_v \cdot x_v$.

The first fundamental form can be written in the matrix form

$$F_1(v,v) = (a,b) \cdot \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix},$$

and can be represented by the matrix

$$\boldsymbol{F}_1 = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}.$$

Definition. Let M be a regular surface in R^3 and let N be a unit normal defined in certain neighborhood of the point $x \in M$. Weingarten mapping is a linear mapping defined by the formula

$$W(v) = -N_v,$$

where $N_v \in T_x(M)$ and N_v is the derivative of N in the direction v.

From the previous definition follows

$$W(u) = -N_u \qquad \text{and} \qquad W(v) = -N_v.$$

So we have

$$\begin{aligned} W(x_u)\cdot x_u &= -N_u \cdot x_u, \qquad W(x_u)\cdot x_v = -N_u \cdot x_v, \\ W(x_v)\cdot x_u &= -N_v \cdot x_u, \qquad W(x_v)\cdot x_v = -N_v \cdot x_v. \end{aligned} \tag{2}$$

Formula (1) gives

$$\begin{aligned} (N\cdot x_u)_u &= N_u \cdot x_u + N\cdot x_{uu} = 0, \\ (N\cdot x_u)_v &= N_v \cdot x_u + N\cdot x_{uv} = 0, \\ (N\cdot x_v)_u &= N_u \cdot x_v + N\cdot x_{vu} = 0, \\ (N\cdot x_v)_v &= N_v \cdot x_v + N\cdot x_{vv} = 0. \end{aligned} \tag{3}$$

From (3) follows

$$N\cdot x_{uv} = -N_v \cdot x_u = W(x_v)\cdot x_u,$$

and

$$N\cdot x_{vu} = -N_u \cdot x_v = W(x_u)\cdot x_v,$$

which means

$$W(x_v)\cdot x_u = W(x_u)\cdot x_v. \tag{4}$$

As x_u and x_v is the basis of $T_x(M)$ we have

$$-N_u = a_{11}x_u + a_{12}x_v, \tag{5}$$

$$-N_v = a_{21}x_u + a_{22}x_v.$$

From (2) and (3) follows

$$\begin{aligned} L_{11} &= W(x_u)\cdot x_u = -N_u \cdot x_u = a_{11}\cdot g_{11} + a_{12}\cdot g_{12}, \\ L_{12} &= W(x_u)\cdot x_v = -N_u \cdot x_v = a_{11}\cdot g_{12} + a_{12}\cdot g_{22}, \\ L_{21} &= W(x_v)\cdot x_u = -N_v \cdot x_u = a_{21}\cdot g_{11} + a_{22}\cdot g_{12}, \\ L_{22} &= W(x_v)\cdot x_v = -N_v \cdot x_v = a_{21}\cdot g_{12} + a_{22}\cdot g_{22}, \end{aligned} \tag{6}$$

where $L_{11} = W(x_u)\cdot x_u$, $L_{12} = W(x_u)\cdot x_v$, $L_{21} = W(x_v)\cdot x_u$ and $L_{22} = W(x_v)\cdot x_v$.

The equation (4) gives $L_{12} = L_{21}$.

The equation (6) can be written in the form

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{12} & L_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}.$$

If we denote $\boldsymbol{L} = \begin{pmatrix} L_{11} & L_{12} \\ L_{12} & L_{22} \end{pmatrix}$, $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $\boldsymbol{G} = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}$, we obtain

$$\boldsymbol{L} = \boldsymbol{A}\cdot\boldsymbol{G} \quad \text{or} \quad \boldsymbol{L}\cdot\boldsymbol{G}^{-1} = \boldsymbol{A}, \tag{7}$$

where

$$\boldsymbol{G}^{-1} = \frac{1}{g_{11}g_{22} - g_{12}^2} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix}.$$

Remark 2. The second fundamental form of surface is

$$F_2(w_1, w_2) = W(w_1)\cdot w_2, \quad \text{where vectors} \quad w_1, w_2 \in T_u(M),$$

$$F_2(w, w) = W(w)\cdot w = W(ax_u + bx_v)\cdot(ax_u + bx_v).$$

Thanks to linearity of W we have

$$\begin{aligned} F_2(u,u) &= (aW(x_u) + bW(x_v))\cdot(ax_u + bx_v) = \\ &= a^2W(x_u)\cdot x_u + abW(x_v)\cdot x_u + abW(x_u)\cdot x_v + b^2W(x_v)\cdot x_v = \\ &= a^2(-N_u\cdot x_u) + ab(-N_v\cdot x_u) + ab(-N_u\cdot x_v) + b^2(-N_v\cdot x_v). \end{aligned}$$

So we have

$$F_2(u,v) = a^2W(x_u)\cdot x_u + 2abW(x_u)\cdot x_v + b^2W(x_v)\cdot x_v = a^2L_{11} + 2ab\,L_{12} + b^2L_{22}.$$

The second fundamental form can be expressed in matrix form

$$F_2(u,u) = (a,b)\cdot\begin{pmatrix} L_{11} & L_{12} \\ L_{12} & L_{22} \end{pmatrix}\cdot\begin{pmatrix} a \\ b \end{pmatrix}.$$

The second fundamental form can be represented by the matrix

$$\boldsymbol{F}_2 = \begin{pmatrix} L_{11} & L_{12} \\ L_{12} & L_{22} \end{pmatrix}.$$

The equation (7) gives

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{1}{g_{11}g_{22} - g_{12}^2}\begin{pmatrix} L_{11} & L_{12} \\ L_{12} & L_{22} \end{pmatrix}\cdot\begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix}.$$

After a short calculation we obtain

$$a_{11} = \frac{L_{11}g_{22} - L_{12}g_{12}}{g_{11}g_{22} - g_{12}^2}, \qquad a_{12} = \frac{-L_{11}g_{12} + L_{12}g_{11}}{g_{11}g_{22} - g_{12}^2},$$
$$a_{21} = \frac{L_{12}g_{22} - L_{22}g_{12}}{g_{11}g_{22} - g_{12}^2}, \qquad a_{22} = \frac{-L_{12}g_{12} + L_{22}g_{11}}{g_{11}g_{22} - g_{12}^2}. \tag{8}$$

From equations (8) follows, that Weingarten mapping can be represented by the matrix

$$\boldsymbol{W} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Mean curvature is

$$\boldsymbol{H} = \frac{1}{2} tr\ \boldsymbol{W} = \frac{1}{2} \frac{L_{11}g_{22} - 2L_{12}g_{12} + L_{22}g_{11}}{g_{11}g_{22} - g_{12}^2}$$

and Gaussian curvature is

$$\boldsymbol{K} = det\ \boldsymbol{W} = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2} = \frac{\det F_2}{\det F_1}.$$

Example 1. Local parameterization of sphere S^2 is

$$x(u,v) \to (r\cos v\cos u, r\cos v\sin u, r\sin v) \quad \text{where} \quad (u,v) \in (0,2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}).$$

The tangent vectors x_u and x_v are

$$x_u = (-r\cos v\sin u, r\cos v\cos u, 0) = r\cos v(-\sin u, \cos u, 0),$$
$$x_v = (-r\sin v\cos u, -r\sin v\sin u, r\cos v) = r(-\sin v\cos u, -\sin v\sin u, \cos v),$$
$$n = r^2(\cos u\cos v, \sin u\cos v, \sin v)$$

and the unit normal

$$N = (\cos u\cos v, \sin u\cos v, \sin v).$$

We have

$$N_u = (-\sin u\cos v, \cos u\cos v, 0),$$
$$N_v = (-\cos u\sin v, -\sin u\sin v, \cos v).$$

From (6) follows

$$-N_u \cdot x_u = a_{11} \cdot g_{11} + a_{12} \cdot g_{12}, \tag{9}$$
$$-N_u \cdot x_v = a_{11} \cdot g_{12} + a_{12} \cdot g_{22},$$

$$-N_v \cdot x_u = a_{21} \cdot g_{11} + a_{22} \cdot g_{12}, \tag{10}$$
$$-N_v \cdot x_v = a_{21} \cdot g_{12} + a_{22} \cdot g_{22},$$

Substituting into (9) we obtain

$$N_u \cdot x_u = (-\sin u \cos v, \cos u \cos v, 0) \cdot (-r \cos v \sin u, r \cos v \cos u, 0) = r \cos^2 v$$

and

$$r\cos^2 v = a_{11} r^2 \cos^2 v + a_{12} r^2 \cos v (\sin v \sin u \cos u - \sin v \sin u \cos u) = a_{11} r^2 \cos^2 v,$$

$$N_u \cdot x_u = r \cos^2 v, \quad x_u \cdot x_v = 0,$$

$$N_u \cdot x_v = 0 = a_{11} \cdot 0 + a_{12} \cdot r^2,$$

from which follows $a_{12} = 0$. So we have

$$r \cos^2 v = a_{11} r^2 \cos^2 v \Rightarrow a_{11} = \frac{1}{r}.$$

Further we have

$$N_v \cdot x_u = 0 \quad \text{and} \quad N_v \cdot x_v = r.$$

From (10) and from the equations

$$N_u \cdot x_v = 0, \quad x_v \cdot x_v = r^2,$$

$$0 = a_{21} \cdot r \cdot \cos^2 v + a_{22} \cdot 0,$$

$$r = a_{21} \cdot 0 \qquad + a_{22} \cdot r^2,$$

follows $a_{21} = 0$ and $a_{22} = \frac{1}{r}$.

Weingarten mapping $W(x_u) = -\partial_u N$ and $W(x_v) = -\partial_v N$ can be represented by the matrix

$$W = \begin{pmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix} = \begin{pmatrix} -\frac{1}{r} & 0 \\ 0 & -\frac{1}{r} \end{pmatrix}.$$

For the Gaussian curvature $\boldsymbol{K} = \det W$ and Mean curvature $\boldsymbol{H} = \frac{1}{2} tr\, W$, we have $\boldsymbol{K} = \frac{1}{r^2}$ as was given in (3) and $\boldsymbol{H} = -\frac{1}{r}$.

Example 2. Torus $T^2 \subset \mathrm{R}^3$: Local parameterization of the torus in R^3 is given by the map

$$x(u,v) = ((a + b\cos v)\cos u, (a + b\cos v)\sin u, b\sin v),$$

where $a > b > 0$, $u \in (0, 2\pi >$ and $v \in (0, 2\pi >$.

The moving frame has the form

$$x_u = (a + b\cos v)\cdot(-\sin u, \cos u, 0),$$
$$x_v = b(-\sin v\cos u, -\sin v\sin u, \cos v),$$
$$N = (\cos u\cos v, \sin u\cos v, \sin v).$$

N is unit normal,

$$-N_u = (\sin u\cos v, -\cos u\cos v, 0),$$

$$N_v = b(\cos u\sin v, \sin v\sin u, -\cos v).$$

Substituting into (9) we obtain

$$-N_u \cdot x_u = (a + b\cos v)\left[-\sin^2 u\cos v - \cos^2 u\cos v\right] = -(a + b\cos v)\cos v$$

$$-N_u \cdot x_v = 0 = a_{11}\cdot 0 + a_{12}\cdot b^2\left[\sin^2 v\cos^2 u + \sin^2 v\sin^2 u + \cos^2 v\right] = a_{12}\cdot b^2 \ ,$$

which means that $a_{12} = 0$. Further we have

$$-N_v \cdot x_v = b^2\left[-\cos^2 u\sin^2 v - \sin^2 v\sin^2 u - \cos^2 v\right] = -b^2 \ .$$

Substituting into (9) we obtain

$$-(a + b\cos v)\cos v = a_{11}(a + b\cos v)^2 + a_{12}(a + b\cos v)\cdot b\cdot\left[\sin v\sin u\cos u - \sin v\sin u\cos u\right],$$
$$-b = a_{21}(a + b\cos v)\cdot\left[\sin v\sin u\cos u - \sin v\sin u\cos u\right]$$
$$+ a_{22}b^2\left[\sin^2 v\cos^2 u + \sin^2 v\sin^2 u + \cos^2 v\right].$$

Previous equations give

$$\cos v = a_{11}(a + b\cos v),$$

$$-b = a_{22}b^2,$$

from which follows $a_{11} = -\dfrac{\cos v}{a + b\cos v}$, $a_{12} = 0$ and $a_{22} = -\dfrac{1}{b}$. Analogically

$$N_v \cdot x_u = -(a + b\cos v)(-\sin u\cos u\sin v + \cos u\sin u\sin v + 0) = 0,$$

$$-N_u \cdot x_u = 0,$$

$$N_v \cdot x_v = b(-\cos u\sin v\sin v\cos u - \sin^2 u\sin^2 v - \cos^2 v) \ ,$$

$$N_v \cdot x_v = -b.$$

Substituting into (10) we obtain

$$0 = a_{21}(a + b\cos v)^2 + a_{22}(a + b\cos v)\cdot(\sin u\cos u\sin v - \sin u\cos u\sin v) \Rightarrow a_{21} = 0,$$
$$-b = a_{21}b(a + b\cos v)(\sin u\cos u\sin v - \sin u\cos u\sin v) +$$
$$+ a_{22}b^2\left(\sin^2 v\cos^2 u + \sin^2 v\sin^2 u + \cos^2 v\right) \Rightarrow a_{22} = -\frac{1}{b}.$$

Weingarten map can be represented by the matrix

$$W = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

which can be written in the form

$$W = \begin{pmatrix} -\dfrac{\cos v}{a + b\cos v} & 0 \\ 0 & -\dfrac{1}{b} \end{pmatrix}.$$

The Gaussian curvature is $\boldsymbol{K} = \det W = \dfrac{\cos v}{b\left(a + b\cos v\right)}$ and Mean curvature H is

$$\boldsymbol{H} = \frac{1}{2} tr\, W = -\frac{1}{2}\left[\frac{\cos v}{a + b\cos v} + \frac{1}{b}\right] = -\frac{1}{2}\left[\frac{b\cos v + a + b\cos v}{b(a + b\cos v)}\right] = -\frac{a + 2b\cos v}{2b(a + b\cos v)}.$$

Example 3. Whitney umbrella:

Local parameterization of this surface is $x(u,v) = (uv, u, v^2)$. Moving frame has the form

$$x_u = (v, 1, 0),$$

$$x_v = (u, 0, 2v),$$

$$n = \left(2v, -2v^2, -u\right).$$

The unit normal is $N = \left(\dfrac{2v}{\sqrt{u^2 + 4v^2 + 4v^4}}, \dfrac{-2v^2}{\sqrt{u^2 + 4v^2 + 4v^4}}, -\dfrac{u}{\sqrt{u^2 + 4v^2 + 4v^4}}\right).$

From previous formula follows

$$N_u = \left(\frac{-2uv}{\left(u^2 + 4v^2 + 4v^4\right)^{\frac{3}{2}}}, \frac{2uv^2}{\left(u^2 + 4v^2 + 4v^4\right)^{\frac{3}{2}}}, -\frac{4v^2 + 4v^4}{\left(u^2 + 4v^2 + 4v^4\right)^{\frac{3}{2}}}\right),$$

$$N_v = \left(\frac{2u^2 - 8v^4}{\left(u^2 + 4v^2 + 4v^4\right)^{\frac{3}{2}}}, \frac{-4u^2 v - 8v^3}{\left(u^2 + 4v^2 + 4v^4\right)^{\frac{3}{2}}}, \frac{4uv + 8uv^3}{\left(u^2 + 4v^2 + 4v^4\right)^{\frac{3}{2}}}\right),$$

$$N_u \cdot x_u = 0, \; -N_u \cdot x_v = \frac{2v}{\sqrt{u^2 + 4v^2 + 4v^4}}, \quad x_u \cdot x_u = v^2 + 1, \quad x_u \cdot x_v = uv,$$

$$-N_v \cdot x_u = \frac{2v}{\sqrt{u^2 + 4v^2 + 4v^4}}, \quad N_v \cdot x_v = \frac{2u}{\sqrt{u^2 + 4v^2 + 4v^4}}, \quad x_v \cdot x_v = u^2 + 4v^2.$$

From (9) follows

$$0 = a_{11}\left(v^2 + 1\right) + a_{12} \cdot uv,$$

$$\frac{2v}{\sqrt{u^2 + 4v^2 + 4v^4}} = a_{11} \cdot uv + a_{12}(u^2 + 4v^2).$$

Using Cramer's rule, we obtain

$$D = \begin{vmatrix} v^2 + 1 & uv \\ uv & u^2 + 4v^2 \end{vmatrix} = u^2 + 4v^2 + 4v^4,$$

$$D_{11} = \begin{vmatrix} 0 & uv \\ \dfrac{2v}{\sqrt{u^2 + 4v^2 + 4v^4}} & u^2 + 4v^2 \end{vmatrix} = \frac{-2uv^2}{\sqrt{u^2 + 4v^2 + 4v^4}},$$

and

$$a_{11} = \frac{D_{11}}{D} = \frac{-2uv^2}{\left(u^2 + 4v^2 + 4v^4\right)^{\frac{3}{2}}}.$$

$$D_{12} = \begin{vmatrix} v^2 + 1 & 0 \\ uv & \dfrac{2v}{\sqrt{u^2 + 4v^2 + 4v^4}} \end{vmatrix} = \frac{2v(v^2 + 1)}{\sqrt{u^2 + 4v^2 + 4v^4}},$$

$$a_{12} = \frac{2v\left(v^2 + 1\right)}{\left(u^2 + 4v^2 + 4v^4\right)^{\frac{3}{2}}}.$$

Analogically we obtain

$$D_{21} = \begin{vmatrix} \dfrac{2v}{\sqrt{u^2 + 4v^2 + 4v^4}} & uv \\ -\dfrac{2u}{\sqrt{u^2 + 4v^2 + 4v^4}} & u^2 + 4v^4 \end{vmatrix} = \frac{2v(2u^2 + 4v^2)}{\sqrt{u^2 + 4v^2 + 4v^4}},$$

$$D_{22} = \begin{vmatrix} v^2 + 1 & \dfrac{2v}{\sqrt{u^2 + 4v^2 + 4v^4}} \\ uv & -\dfrac{2u}{\sqrt{u^2 + 4v^2 + 4v^4}} \end{vmatrix} = \frac{2u(1 + 2v^2)}{\sqrt{u^2 + 4v^2 + 4v^4}},$$

and consequentially we have

$$a_{21} = \frac{D_{21}}{D} = \frac{4v\left(u^2 + 2v^2\right)}{\left(u^2 + 4v^2 + 4v^4\right)^{\frac{3}{2}}},$$

$$a_{22} = \frac{D_{22}}{D} = \frac{-2u\left(1+2v^2\right)}{\left(u^2+4v^2+4v^4\right)^{\frac{3}{2}}}.$$

Finally, the matrix represented Weingarten map has the form

$$W = \begin{vmatrix} -\dfrac{2uv^2}{\left(u^2+4v^2+4v^4\right)^{\frac{3}{2}}} & \dfrac{2v(v^2+1)}{\left(u^2+4v^2+4v^4\right)^{\frac{3}{2}}} \\ \dfrac{4v\left(u^2+2v^2\right)}{\left(u^2+4v^2+4v^4\right)^{\frac{3}{2}}} & \dfrac{-2u\left(1+2v^2\right)}{\left(u^2+4v^2+4v^4\right)^{\frac{3}{2}}} \end{vmatrix}.$$

Gaussian curvature $\boldsymbol{K} = \det W = \dfrac{-4v^2}{(u^2+4v^2+4v^4)^2}$ and the formula for Mean curvature is

$$\boldsymbol{H} = \frac{1}{2} tr\, W = -\frac{u+3uv^2}{\left(u^2+4v^2+4v^4\right)^{\frac{3}{2}}}.$$

Example 4. Hyperbolical paraboloid.

Local parametrization of this surface is $x(u,v) = (u,v,uv)$ and moving frame has the form

$$x_u = (1,0,v), \quad x_v = (0,1,u), \quad n = (-v,-u,1), \quad N = \frac{n}{\|n\|},$$

$$N = \left(\frac{-v}{\sqrt{1+u^2+v^2}}, \frac{-u}{\sqrt{1+u^2+v^2}}, \frac{1}{\sqrt{1+u^2+v^2}} \right).$$

The Weingarten map gives

$$N_u \in T_x(M) \text{ and } N_v \in T_x(M),$$

where

$$-N_u = \frac{\left(-uv, 1+v^2, u\right)}{\left(1+u^2+v^2\right)^{\frac{3}{2}}},$$

$$-N_v = \frac{\left(1+u^2, -uv, v\right)}{\left(1+u^2+v^2\right)^{\frac{3}{2}}}.$$

From (5) follows

$$-N_u \cdot x_u = a_{11}(1+v^2) + a_{12}(uv),$$

$$-N_u \cdot x_v = a_{11}(uv) + a_{12}(1+u^2).$$

Further we have

$$-N_u \cdot x_u = 0, \quad -N_u \cdot x_v = \frac{1+u^2+v^2}{\left(1+u^2+v^2\right)^{\frac{3}{2}}}.$$

The system of equations

$$0 = a_{11}(1+v^2)+a_{12}(uv),$$

$$\frac{1+u^2+v^2}{\left(1+u^2+v^2\right)^{\frac{3}{2}}} = a_{11}\left(uv\right)+a_{12}\left(1+u^2\right). \qquad (11)$$

Using the Cramer's rule we obtain

$$D = (1+v^2)(1+u^2)-u^2v^2 = 1+u^2+v^2,$$

$$a_{11} = \frac{D_{11}}{D} = \frac{det\begin{pmatrix} 0 & uv \\ \frac{1+u^2+v^2}{\left(1+u^2+v^2\right)^{\frac{3}{2}}} & 1+u^2 \end{pmatrix}}{1+u^2+v^2} = -\frac{uv}{\left(1+u^2+v^2\right)^{\frac{3}{2}}},$$

$$a_{12} = \frac{D_{12}}{D} = \frac{\det\begin{pmatrix} 1+v^2 & 0 \\ uv & \frac{1+u^2+v^2}{\left(1+u^2+v^2\right)^{\frac{3}{2}}} \end{pmatrix}}{1+u^2+v^2} = \frac{1+v^2}{\left(1+u^2+v^2\right)^{\frac{3}{2}}}.$$

Analogically

$$-N_v = \left(\frac{1+u^2}{\left(1+u^2+v^2\right)^{\frac{3}{2}}}, \frac{-uv}{\left(1+u^2+v^2\right)^{\frac{3}{2}}}, \frac{v}{\left(1+u^2+v^2\right)^{\frac{3}{2}}}\right).$$

$$-N_v = a_{21}x_u + a_{22}x_v,$$

$$-N_v \cdot x_u = a_{21}\, x_u \cdot x_u + a_{22}\, x_v \cdot x_u,$$

$$-N_v \cdot x_v = a_{21}\, x_u \cdot x_v + a_{22}\, x_v \cdot x_v.$$

$$\frac{1+u^2+v^2}{\left(1+u^2+v^2\right)^{\frac{3}{2}}} = a_{21}(1+v^2)+a_{22}(uv),$$

$$0 = a_{21}(uv)+a_{22}(1+u^2).$$

$$D = (1+v^2)(1+u^2) - u^2v^2 = 1+u^2+v^2,$$

$$a_{21} = \frac{det\begin{pmatrix} -\dfrac{1+u^2+v^2}{\left(1+u^2+v^2\right)^{\frac{3}{2}}} & uv \\ 0 & 1+u^2 \end{pmatrix}}{1+u^2+v^2} = \frac{1+u^2}{\left(1+u^2+v^2\right)^{\frac{3}{2}}},$$

$$a_{22} = \frac{det\begin{pmatrix} 1+v^2 & \dfrac{1+u^2+v^2}{\left(1+u^2+v^2\right)^{\frac{3}{2}}} \\ uv & 0 \end{pmatrix}}{1+u^2+v^2} = \frac{-uv}{\left(1+u^2+v^2\right)^{\frac{3}{2}}}.$$

So the Weingarten matrix W has the form

$$W = \begin{pmatrix} \dfrac{-uv}{\left(1+u^2+v^2\right)^{\frac{3}{2}}} & \dfrac{1+v^2}{\left(1+u^2+v^2\right)^{\frac{3}{2}}} \\ \dfrac{1+u^2}{\left(1+u^2+v^2\right)^{\frac{3}{2}}} & \dfrac{-uv}{\left(1+u^2+v^2\right)^{\frac{3}{2}}} \end{pmatrix}.$$

Further we have

$$\boldsymbol{K} = det\, W = \frac{1}{\left(1+u^2+v^2\right)^3}\left[u^2v^2 - \left(1+u^2 + v^2 - u^2v^2\right)\right] = \frac{-1}{\left(1+u^2+v^2\right)^2},$$

and

$$\boldsymbol{H} = \frac{1}{2} tr\, W = \frac{-uv}{\left(1+u^2+v^2\right)^{\frac{3}{2}}}.$$

Conclusion

Gauss and Mean curvature in studied surfaces are:

1. Sphere $\boldsymbol{K} = \frac{1}{r^2}$ and $\boldsymbol{H} = -\frac{1}{r}$.
2. Torus $\boldsymbol{K} = \frac{\cos v}{b(a+b\cos v)}$ and $\boldsymbol{H} = -\frac{a+2b\cos v}{2b\left(a+b\cos v\right)}$.
3. Whitney umbrella $\boldsymbol{K} = \frac{-4v^2}{u^2+4v^2+4v^4}$. and $\boldsymbol{H} = -\frac{u+3uv^2}{\left(u^2+4v^2+4v^4\right)^{\frac{3}{2}}}$.
4. Cobb-Douglas surface $\boldsymbol{K} = \frac{-1}{\left(1+u^2+v^2\right)^2}$ and $\boldsymbol{H} = \frac{-uv}{\left(1+u^2+v^2\right)^{\frac{3}{2}}}$.

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