AN APPLICATION OF DISCRETE MATHEMATICAL MODEL IN ECONOMICS

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Abstract

This paper presented a comprehensive mathematical model using the second-order linear difference equation with constant coefficients in an analysis of inventory cycles with study of national income and its progress in time. The solution of this difference equation has the stable equilibrium value. In this "pure inventory cycle" model, expectation of future sales depends only on the level of past sales.

Key words: linear difference equation, homogeneous difference equation, auxiliary equation, equilibrium value, stable equilibrium value, inventory cycle, national income

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Introduction

This paper continues and extends the article Coufal (2011). A type of multiplier-accelerator model was employed by Lloyd A. Metzler (1941) in his famous "inventory cycle". This economic model is concerned with study of national income and its variation in time. Metzler's essential idea was that producers desire to keep inventory stock as some proportion of expected sales but, relying on lags between production and sales similar to that of Erik Lundberg (Metzler, 1941). Metzler contended that the precise inventory policy chosen by producers might have profound effects on the economy – particularly, in generating various different dynamics. He presented a comprehensive mathematical model using difference equations in an analysis of inventory cycles.

1 The mathematical model in an analysis of inventory cycles

We shall outline only the beginning of this theory here. It is an example of *period analysis,* in which time is divided into intervals of equal length (which may be thought of as years) and all functions are dated with the period at which they are to be evaluated.

Entrepreneurs produce consumer goods for two purposes:

a) for sales, and

b) for maintaining certain optimum inventory levels.

With units of measurement appropriately chosen, let us introduce symbols as follows:

u_t is number of units of consumers‘ goods produced for sale on period t, and s_t is number of units of consumers‘ goods produced for inventories on period t.

We assume there is a constant autonomous net investment, denoted by v_0, in each period.

The total income produced in period t is equal to the total production of consumers‘ goods plus net investment. If y_t is total income in period t, then

(1) $$y_t = u_t + s_t + v_0 \quad \text{for} \quad t = 0,\ 1,\ 2,\ \ldots,\ n,\ \ldots$$

Producers planning their output for sales for any period do so at the beginning of that period and base their production plans on the sales of the preceding period. We assume actual sales of any period are a fraction β of the total income of that period. We further assume that planned output for period t is taken to be equal to actual sales of period $t-1$, i.e.,

(2) $$u_t = \beta\ y_{t-1} \quad \text{for} \quad t = 1,\ 2,\ 3,\ \ldots,\ n,\ \ldots$$

The constant of proportionality β in (2) is the *marginal propensity to consume* (of this year's consumption with respect to last year's income). The constant is restricted by the condition

(3) $$0 < \beta < 1.$$

That Keynesian hypothesis is that the marginal propensity to consume is positive but less than unity is of great analytical and practical significance. Besides telling us that consumption is an increasing function of income and it increases by less than the increment of income, this hypothesis helps in explaining

a) the theoretical possibility of general overproduction or 'underemployment equilibrium' and also

b) the relative stability of a highly developed industrial economy. (Barro and Grillio, 1994, Samuelson, 1947, Samuelson and Nordhaus, 2004)

Note that (3) holds for t-values starting with $t = 1$ (rather than $t = 0$), since we are relating variables for period t and the preceding period $t-1$.

Production for inventory stocks during any period is also decided at the beginning of each period and we assume that an attempt is made to maintain inventories at some constant (normal) level. That is, businessmen will ordinarily attempt to replenish inventories depleted by an unforeseen rise in demand, or to reduce inventory accumulations resulting from unpredicted depressions (Asimakopulos, 1983).

To be definite, Metzler assumes that production of goods for inventories in any period t is equal to the difference between actual and anticipated sales of the preceding period $t-1$. Throughout we are assuming adequate inventories are maintained so that all differences between production and consumer demand can be met by inventory fluctuations rather than by price changes.

Now anticipated sales of period $t-1$ are u_{t-1}, which, by (2), is equal to $\beta\ y_{t-2}$. And by an assumption already made, actual sales of period $t-1$ are β times the income of period $t-1$, i.e., actual sales of period $t-1$ are $\beta\ y_{t-1}$. Hence

(4) $$s_t = \beta\ y_{t-1} - \beta\ y_{t-2} \quad \text{for } t = 2, 3, 4, \ldots, n, \ldots$$

It is now possible to derive a difference equation for the net income y. Starting with (1) and using (2) and (4), we obtain

$$y_t = \beta\ y_{t-1} + \beta\ y_{t-1} - \beta\ y_{t-2} + v_0$$

or

(5) $$y_t - 2\beta\ y_{t-1} + \beta\ y_{t-2} = v_0 \quad \text{for } t = 2, 3, 4, \ldots, n, \ldots$$

If we wish to have our difference equation defined over the set of t-values beginning with $t = 0$, we rewrite (5) in the equivalent form

(6) $$y_{t+2} - 2\beta\ y_{t+1} + \beta\ y_t = v_0 \quad \text{for } t = 0, 1, 2, \ldots, n, \ldots$$

We recognize (6) as second-order linear difference equation with constant coefficients. (Cull, Flahive and Robson, 2005 – chapter 7, Jacques 2006 – chapter 9.1 *Difference Equations,* 551–568)

2 The solution

We now are able to complete the discussion initiated there of Metzler's "pure inventory cycle." We now find the solution of (6) and show that the sequence $(y_t)_{t=0}^{\infty}$ of income values will undergo damped oscillatory behavior around a limit determined by the values of β and v_0.

The homogeneous difference equation corresponding to (6) is (Cull, Flahive and Robson, 2005 – chapter 7, Jacques 2006 – chapter 9.1 *Difference Equations,* 551–568)

(7) $$y_{t+2} - 2\beta\ y_{t+1} + \beta\ y_t = 0 \quad \text{for} \quad t = 0,\ 1,\ 2,\ \ldots,\ n,\ \ldots$$

and the auxiliary equation to (7) is (Cull, Flahive and Robson, 2005 – chapter 7, Jacques 2006 – chapter 9.1 *Difference Equations,* 551–568)

(8) $$\lambda^2 - 2\beta\ \lambda + \beta = 0\,.$$

Since (3), discriminant $D = 4\beta^2 - 4\beta = 4\beta(\beta - 1) < 0$, than the roots the auxiliary equation (8) are the complex numbers

$$\lambda_{1,2} = \frac{2\ \beta\ \pm\ i\ 2\ \sqrt{\beta \cdot (1-\beta)}}{2} = \beta\ \pm\ i\ \sqrt{\beta \cdot (1-\beta)}\,.$$

We write these complex conjugate roots in polar form. Their absolute value r is given

(9) $$r = |\lambda_1| = |\lambda_2| = \sqrt{\beta^2 + \beta \cdot (1-\beta)} = \sqrt{\beta^2 + \beta - \beta^2} = \sqrt{\beta}$$

and argument φ is angle between 0 and $\frac{\pi}{2}$ (i.e. $\varphi \in \left(0,\ \frac{\pi}{2}\right)$) for which

$$\cos\varphi = \frac{\beta}{r} = \frac{\beta}{\sqrt{\beta}} = \sqrt{\beta} \ \wedge\ \sin\varphi = \frac{\sqrt{\beta \cdot (1-\beta)}}{r} = \frac{\sqrt{\beta(1-\beta)}}{\sqrt{\beta}} = \sqrt{1-\beta}$$

(since (3), $\cos\varphi > 0 \wedge \sin\varphi > 0$).

Then the general solution of the homogeneous equation (7) is

$$\bar{y}_t = C_1(\sqrt{\beta})^t \cos(t\varphi) + C_2(\sqrt{\beta})^t \sin(t\varphi)\,,$$

where C_1 and C_2 are arbitrary real constants.

To find a particular solution of (6) we use a trial solution of the form $y_t^* = k$, a constant. If this to satisfy (6), we must have

$$k - 2\beta\ k + \beta\ k = v_0\,,\ \text{i. e. } k = \frac{v_0}{1-\beta}$$

so a particular solution is given by

(10) $$y_t^* = \frac{v_0}{1-\beta}\,.$$

Thus, the general solution (y_t) of the national income equation (6) has the form

(11) $$y_t = \bar{y}_t + y_t^* = C_1(\sqrt{\beta})^t \cos(t\varphi) + C_2(\sqrt{\beta})^t \sin(t\varphi) + \frac{v_0}{1-\beta}.$$

The cosine and sine terms produce cyclical fluctuations of its oscillation between positive a negative values. These fluctuations are damped by the factor $(\sqrt{\beta})^t$ since $0 < \beta < 1$.

3 Equilibrium and Stability

The nature of the solution of the solution of a linear difference equation with constant coefficients, especially its limiting behavior, is dependent upon the initial values prescribed for the solution and the roots of the auxiliary equation.

Consider the nth order linear difference equation with constant coefficients

(12) $y_{t+n} + a_1 \cdot y_{t+n-1} + a_2 \cdot y_{t+n-2} + ... + a_{n-1} \cdot y_{t+1} + a_n \cdot y_t = b$ for $t = 0, 1, 2, \dots, k, ...$

where $a_1, a_2, ..., a_{n-1}, a_n$ are real numbers, $a_n \neq 0$. We write b in place usual b_t since we now assume a constant right-hand term.

The homogeneous difference equation corresponding to (12) is

(13) $$y_{t+n} + a_1 \cdot y_{t+n-1} + a_2 \cdot y_{t+n-2} + ... + a_{n-1} \cdot y_{t+1} + a_n \cdot y_t = 0$$

and the auxiliary equation to (13) is

(14) $$\lambda^n + a_1 \cdot \lambda^{n-1} + a_2 \cdot \lambda^{n-2} + ... + a_{n-1} \cdot \lambda + a_n = 0.$$

If (12) has a constant sequence as solution, then value of this sequence is called an *equilibrium value*[1] of this difference equation (12) or (y_t).

Putting $y_t = y^*$, a constant, for $t = 0, 1, 2, \dots, k, ...$ in (12), we find

$$y^* + a_1 \cdot y^* + a_2 \cdot y^* + ... + a_{n-1} \cdot y^* + a_n \cdot y^* = b.$$

Hence, if $1 + a_1 + a_2 + ... + a_{n-1} + a_n \neq 0$ (i. e. if and only if number 1 is not root the auxiliary equation (14)), then

(15) $$y^* = \frac{b}{1 + a_1 + a_2 + ... + a_{n-1} + a_n}$$

is equilibrium of (y_t). Such an equilibrium value has the following property: if n consecutive values of any solution (y_t) of (12) are equal to y^*, then all succeeding values of (y_t) are equal to y^*.

[1] *Stationary value* or *fixed point*

The equilibrium value y^* *is said to be stable*[2] *if every solution* (y_t) *of the equation* (12), *independently of the prescribed initial conditions* $y_0,\ y_1,\ ...,\ y_{n-1}$, *converges to* y^*, *i. e., if*

$$\lim_{t\to\infty} y_t = y^*$$

for all $y_0,\ y_1,\ ...,\ y_{n-1}$.

Since a displacement from the equilibrium value is equivalent to considering a new solution with different initial conditions, we may alternatively define a stable equilibrium as one for which any displacement from equilibrium is followed by a sequence of values of (y_t) which again converge to this equilibrium.

It is convenient to define a new sequence (z_t) which measures the deviation of a solution (y_t) of (12) from its equilibrium value y^*; i. e., we let

$$z_t = y_t - y^*.$$

Then

$$\begin{aligned} &z_{t+n} + a_1 \cdot z_{t+n-1} + a_2 \cdot z_{t+n-2} + ... + a_{n-1} \cdot z_{t+1} + a_n \cdot z_t = \\ &= (y_{t+n} + a_1 \cdot y_{t+n-1} + a_2 \cdot y_{t+n-2} + ... + a_{n-1} \cdot y_{t+1} + a_n \cdot y_t) - (1 + a_1 + a_2 + ... + a_{n-1} + a_n) \cdot y^* = \\ &= y_{t+n} + a_1 \cdot y_{t+n-1} + a_2 \cdot y_{t+n-2} + ... + a_{n-1} \cdot y_{t+1} + a_n \cdot y_t - b = b - b = 0 \end{aligned}$$

since (y_t) is a solution of (12). Hence (z_t) is a solution of the homogeneous difference equation (13). The definition of stability requires that (z_t), deviation of (y_t) from its equilibrium value, converge to 0 for every initial values $z_0,\ z_1,\ ...,\ z_{n-1}$, i. e. $\lim_{t\to\infty} z_t = 0$

for all $z_0,\ z_1,\ ...,\ z_{n-1}$. Since (z_t) is a solution of (13), this result supplies an immediate proof of the following theorem.

Theorem. *A necessary and sufficient condition for the equilibrium value* y^* *in* (15) *to be stable is* $\mu < 1$, *where*

$$\mu = \max\{|\lambda_i|\ ;\ i = 1,\ 2,\ ...,\ k\}$$

and $\lambda_1,\ \lambda_2, ...,\ \lambda_k$ *are all roots of the auxiliary equation* (14).

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[2] Or the difference equation (12) is *stable.* A number of different definitions of stability appear in the literature. We here define what is know as "perfect stability of the first kind." (Samuelson, 1947, p. 261)

Conclusion

Return to Metzler‘s difference equation (6) which we repeat here:

$y_{t+2} - 2\beta \ y_{t+1} + \beta \ y_t = v_0$ for $t = 0,\ 1,\ 2,\ \dots,\ n,\ \dots$

Recall that y_t denotes the total income in period t and β is the *marginal propensity to consume*. We assume $0 < \beta < 1$.

An equilibrium value of the total income is $y^* = \dfrac{v_0}{1-\beta}$ (see (10)).

The stability condition $\mu < 1$ when applied to the difference equation (6) become:

$$\mu = \max \ \{|\lambda_1|,\ |\lambda_2|\} = \sqrt{\beta} < 1 \Rightarrow \lim_{t\to\infty} y_t = y^* = \frac{v_0}{1-\beta},$$

than the equilibrium value of (6) is stable.

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