THE DECISIONS ON THE OPTIMAL PORTFOLIO UNDER THE CAPM AND RISK MEASUREMENT

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Abstract

In this paper we will discuss the allocation problem from the perspective of an asset manager or an investment institution. Investors make decision about efficient allocation of their resources. To utilize their resources efficiently they need to balance high return, higher risk activities with those that have low return and lower risk. But how should they choose the 'best' mix of activities? How can a fund manager choose his investments in different assets to optimize the performance of his portfolio? How should he measure the performance of his investments? How should he control the risk of his portfolio? The investors need to use methods that focus on the proper aggregation of risks, taking into account the netting of positions and the correlations between assets and risk factors. Because of these reasons we use copula approach to compute optimal portfolio weight. In this paper we have used the optimization of the portfolio weight based on maximized Generalized Sharpe Ratio. We have also computed Generalized Sharpe Ratio based on Value–at–Risk of our portfolio as a risk measure.

Key words: CAPM, copula, portfolio, Sharpe ratio, Value–at–Risk (VaR)

JEL Code: G11, G12, G32, C80, C58

Introduction

There are a number of approaches that can be used to obtain optimal portfolio and all of the approaches have their pros and cons. Historical returns are not stable, the future does not repeat the past. This is one of the reasons why mathematicians study models that might capture fluctuations of the assets. Note that even using more complex models, fluctuations of the estimates will still exist. They are an eliminable consequence of the global uncertainty in financial markets. The point is that the fluctuation of the estimates should not be too large to validate the model that is assumed (Focardi, 2004).

Von Neumann and Morgenstern (1947) were pioneers in dealing with this problem. They introduced the theory of utility functions and explained the different investor attitudes to risk and have shown how an investor's utility determines his optimal portfolio. In the 1950s Harry Markowitz considered solving the portfolio allocation decision as a "risk averse" investor and introduced the principle of portfolio diversification and the minimum variance portfolio allocation problem, with and without constraints on allocations. Markowitz's work laid the foundation for the development of the *theory of asset pricing*. In the sixties Treynor (1965), Sharpe (1964) and Lintner (1965) have independently developed CAPM. CAPM introduces the concept of the market beta of an asset, also called its systematic risk. The model implies that assets with no systematic risk must earn the risk free rate, and any excess return over the risk free rate is proportional to the systematic risk. The market beta is derived from the covariance of the asset return and the market portfolio return. However, the covariance is just the first moment of the joint distribution between the returns, and this may be estimated using only recent historical data – or indeed, it could be set without using historical data at all – it could be based on the personal view of the portfolio manager. However, the assumption of multivariate normal asset returns is not very realistic, and this approach can lead to portfolio allocations that are very unstable over the time (Alexander, 2008).

The aim of this paper is to compute optimal portfolio weight under maximal Generalized Sharpe Ratios using Normal and Student *t* copula and to advise risk and return analysis. We have analysed the decision rule proposed by Dowd (2000) and Tasche (2001). It enables a manager to assess alternative investment opportunities, where the alternatives have different expected pay–off and risks.

The paper is organized as follows. Section 1 assesses the traditional or standard Sharpe ratios approach. If the returns are normal and the investments being considered are independent of the rest of our portfolio, but cannot be relied upon otherwise, the standard Sharpe ratio gives the correct result. Section 2 introduces copula approach to portfolio allocation problem. This section gives outline how a multivariate copula can be used in portfolio optimization with the aim of increasing the stability of optimal allocations. Section 3 presents the case study portfolio of two assets: gold in USD and FX rate GBP/USD. The results were obtained by Wolfram Mathematica 8.2 software. Conclusion summarizes the results.

1 The generalized Sharpe decision rule

Let us consider a portfolio with random return X over the risk–free interest rate. Suppose that the financial market offers the possibility to invest in a new asset with random return Y over the risk–free rate. We assume that both X and Y are integrable, i.e.

$$\mathrm{E}[X] < \infty \text{ and } \mathrm{E}[Y] < \infty \tag{1.1}$$

If necessary, we also assume the existence of higher moments of all involved random variables. Denote the expected returns by $R_X = \mathrm{E}[X]$ and $R_Y = \mathrm{E}[Y]$ respectively. Let $w \in [0, 1]$ be the relative weight of Y in a pooled portfolio consisting of X and Y. The random return to the pooled portfolio is then

$$Z(w) = (1-w)X + w\,Y \tag{1.2}$$

Denote by $R(w)$ the expected return to the pooled portfolio, i.e.

$$R(w) = \mathrm{E}[Z(w)] = (1-w)\,R_X + w\,R_Y. \tag{1.3}$$

Let $\rho(w)$ denote a risk measure of the pooled portfolio (i.e. $\rho(0)$ is the risk of the existing portfolio) and $S(w)$ denote the *Generalized Sharpe Ratio* of the portfolio, i.e.

$$S(w) = \frac{R(w)}{\rho(w)} \tag{1.4}$$

In the special case $\rho(w)$ that coincides with the standard deviation, the classical Sharpe Ratio comes out. As in the traditional mean–risk approach, the portfolio $Z(w)$ is preferred to $Z(0)$ if

$$S(w) > S(0) \text{ for some } w, \tag{1.5}$$

i.e., the trade–off between expected return and risk is more favourable than that provided by the current portfolio $Z(0) = X$. Hence, the relative weight of the total investment is shifted from X to Y. We call the decision rule based on (1.5) the Generalized Sharpe decision rule (Tasche, Tibiletti, 2001).

In the following we consider two popular definitions of risk measures. Define for fixed $\alpha \in (0,1)$ the α–quantile q_α of a random variable Φ by

$$q_\alpha(\Phi) = \mathit{inf}\{\, \varphi \in \mathbf{R} : P[\Phi \le \varphi] \ge \alpha \}, \tag{1.6}$$

Right now, we are about to consider the following special risk measures:

Definition 1 Let Φ be a real valued random variable. Then

1. its Standard deviation is

$$\rho_1(\Phi) = \sqrt{E[(\Phi - E(\Phi))^2]} \tag{1.7}$$

2. its Value–at–Risk (VaR) at level $\alpha \in (0,1)$ is

$$\rho_2(\Phi) = E[\Phi] - q_\alpha(\Phi)\,. \tag{1.8}$$

We will apply these risk measures to the random return $Z(w)$ of the pooled portfolio:

$$q_i\,(w) = q_i\,(Z(w)), \qquad i = 1,2 \tag{1.9}$$

Note that we use definitions of the risk measure *relative to the mean* in the sense of Jorion, (1997).

The use of VaR as a risk measure is suggested to capture information on the extreme events. Therefore, its main aim is to take under control distributions with “long and thin” left tails.

The Generalized Sharpe ratio captures both risk and return, depending on the circumstances in a single measure. Arising return differential or a falling standard deviation both “good” events leads to a rise in the Sharpe ratio; conversely, a falling return differential or a rising standard deviation of both “bad” events leads to a fall in the Sharpe ratio. Hence, a higher Sharpe ratio is good, and a lower one is bad. When choosing between two alternatives, the Sharpe ratio criterion is therefore to choose the one with the higher Sharpe ratio. (Dowd, 2000).

2 Portfolio optimization using copula

To find the optimal mix of risky assets in a portfolio, portfolio optimizer can apply any of the performance measures described in Section 1 to a distribution of portfolio returns. This distribution can be generated using either (Alexander, 2008)

- simulation of a current weighted time series of portfolio returns, or
- an assumption that the asset returns have a multivariate normal distribution, in which we only need to know the returns covariance matrix.

The advantage of the first approach is that the portfolio returns distribution is based on experienced rather than assumed behaviour of the asset returns. The disadvantage of this approach is that we need to have long time series of returns on each asset, which may not be available. The second approach requires only the asset returns covariance matrix, and this may be estimated using only recent historical data – or indeed, it could be set without using historical data at all – it could be based on the personal view of the portfolio manager.

In this paper we use multivariate copula[1] for portfolio optimization. Optimization will still be based on a correlation matrix, but we are now free to specify for the marginal their empirical distributions. The normality constraints for the marginal are no longer necessary.

[1] The copula is typically used to construct a joint distribution and gives us a functional form that captures the observed behaviour of financial asset returns far better than an elliptical distribution (Sklar, 1959, 1996), (Embrecht, 2001), (Cherubini, 2004), (Alexander, 2008).

Let every return asset X_i, i=1, 2 have a density function (marginal density function of the portfolio). We can translate it into a marginal density of w_iX_i by multiplying the density of X_i by w_i^{-1}, for i=1,2[2]. Then the distribution of the portfolio return $w_1X_1+w_2X_2$ may be derived only from their joint density function $f(x_1, x_2)$, using the convolution integral (2.1). Let $Y=X_1+X_2$ and denote the density of Y by $g(y)$. Then:

$$\boldsymbol{g}(\boldsymbol{y}) = \int_{x_1} \boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{y}-\boldsymbol{x}_1)\boldsymbol{dx}_1 = \int_{x_2} \boldsymbol{f}(\boldsymbol{y}-\boldsymbol{x}_2, \boldsymbol{x}_2)\boldsymbol{dx}_2 . \tag{2.1}$$

In order to derive the distribution of the sum of two random variables, we need to know their joint density function, assuming it exists. If we know the marginal distributions of two returns X_1, X_2 and a copula function, we can obtain the joint density $f(x_{1,}\, x_2)$. Then we apply (2.1) to obtain the density of their sum (see Alexander, 2008).

Such a density will also depend on the copula and the marginal distribution that is used to model the joint distribution of X_1, X_2.

Suppose we are given the marginal densities of the asset returns and a copula. For some fixed set of portfolio weights we construct the density of $w_1X_1 + w_2X_2$, as defined above. From this density we can derive a risk adjusted performance measure of the Sharpe ratio (1.4). Now we can change the portfolio weights, each time using the marginal and the copula to derive the portfolio return density. Hence, we can find the optimal portfolio weights, i.e. these that maximize our performance metric.

3 Case study

In this section, we present case study. We create a portfolio composed of Gold price against USD (AUX/USD) and exchange rate GBP against USD (GBP/USD). Analysed period was from the 2st of January 2008 to the 19st of July 2012 in daily frequency. Our data follows finance.yahoo.com.

The aim of this paper is to create portfolio of the AUX/USD and GBP/USD that maximizes the Generalized Sharpe ratio (1.4) based on risk measure ρ_1 (1.7) and to compute Generalized Sharpe ratio based on ρ_2 (1.8) for fixed set free risk returns.

To determine optimal weight of our portfolio we must proceed as follows:

1. to compute parameter – degree of freedom of the Student t marginal distributions for AUX/USD and GBP/USD,
2. to compute correlation parameter ρ for normal copula with Student t marginal

[2] Since the distribution function of wX is $F(w^{-1}X)$, where $F(X)$ is the distribution of X.

3. to compute parameters (degrees of freedom ν and correlation parameter ρ) for Student copula with Student *t* marginal
4. to use the copula and the marginal distributions to compute the joint density of standardized returns, i.e. returns with zero mean and unit standard deviation,
5. to set fixed risk free return from 1%, 2%, …, 10%
6. for a fixed set of portfolio weights, compute the portfolio return,
7. estimate the Generalized Sharpe ratio using the formula (1.4)
8. to find the portfolio weights that maximize the Generalized Sharpe ratio based on risk measure ρ_1 (1.7).
9. to compute Generalized Sharpe ratio based on risk measure ρ_2 (1.8).

The annual mean returns of the AUX/USD and FX rate GBP/USD during the considered period is 13.639% and 5.035% respectively. Volatilities of returns are 22.669% and 12.173% respectively (see Table 1). Table 1 shows that over the sample period the allocation of AUX/USD has higher annual mean return than GBP/USD and volatility of the AUX/USD is higher than volatility of GBP/USD too. Investing in AUX/USD gives higher return, but also greater risk. Fig.1. shows histogram of daily log returns on AUX/USD and GBP/USD during analysed period. Use these data to calibrate (a) Normal copula and (b) Student *t* copula. In each case assume the marginal are Student *t* distribution. We found degree of freedom 3.899 and 4.987 for AUX/USD and GBP/USD respectively using MLE[3] (see Table 1).

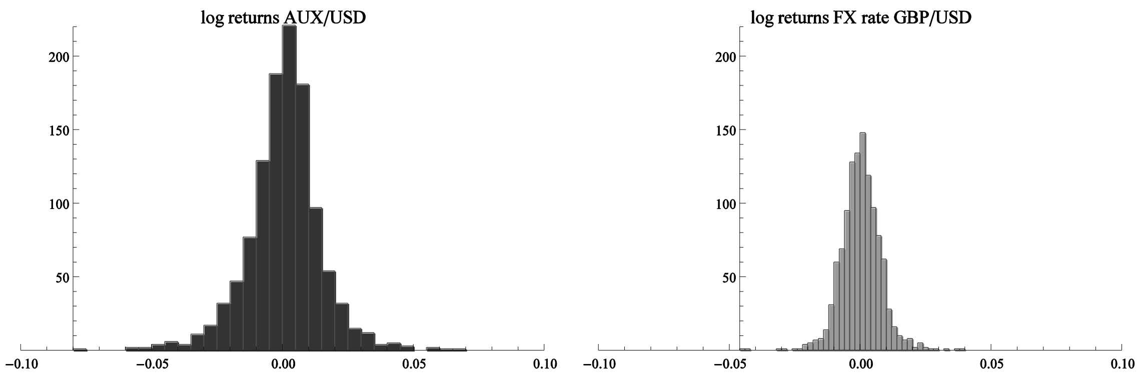
Table 1: Descriptive statistics of the AUX/USD and GBP/USD returns.

	AUX/USD	***GBP/USD***
Mean of returns	0.056%	0.020%
Standard deviation of return	0.769%	1.435%
Annual Mean	13.639%	5.035%
Volatility of return	22.669%	12.173%
Degree of freedom	3.899	4.987

Source: Calculated by the authors with Wolfram Mathematica software on base data from finance.yahoo.com

Fig. 1: Histograms of the AUX/USD and GBP/USD log returns

[3] MLE - Maximum Likelihood Method is general method for estimating the parameters of a distribution.



Source: Calculated by the authors with Wolfram Mathematica software on base data from finance.yahoo.com

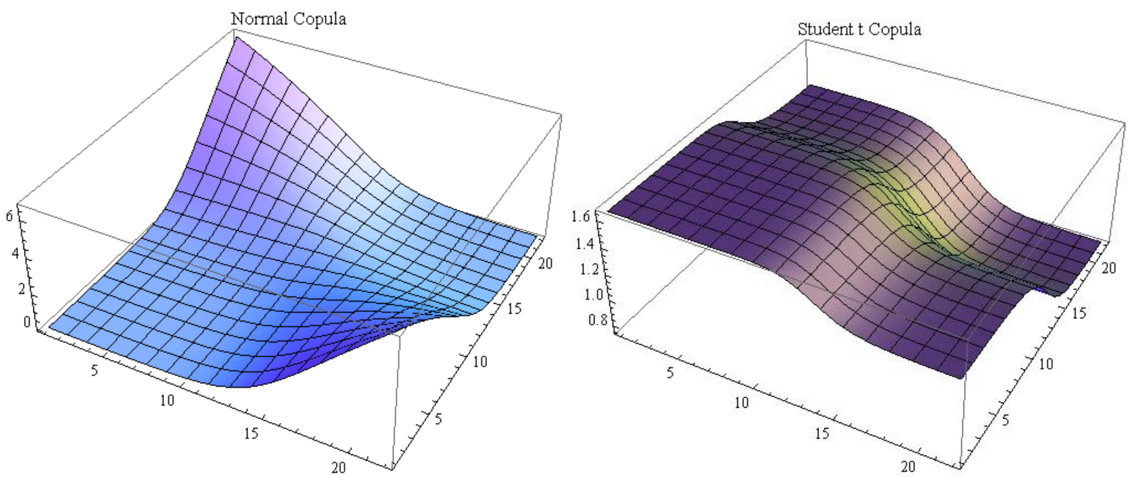
Now we calibrate the copula parameters. First, consider Normal copula case (a). Normal copula has one parameter: the correlation ρ. We calibrate this parameter by using its relationship with a rank Spearman correlation (Alexander, 2008). Secondly, we calibrate Student *t* copula, case (b). Bivariate Student *t* copula has two parameters: the correlation ρ and the degrees of freedom ν. We calibrate both ρ and ν simultaneously using MLE. Table 2 shows calibrated parameters for both cases (a) and (b). These copula functions we can see on Fig.2. Finally, we compute joint density function for both cases (a) and (b) (see Fig. 3).

Table 2: Calibrated parameters of copulas

Parameters of copulas	
Correlation parameter for Normal copula	−0.3126
Correlation parameter for Student t copula	−0.4434
Degree of freedom of Student t copula	6.6571

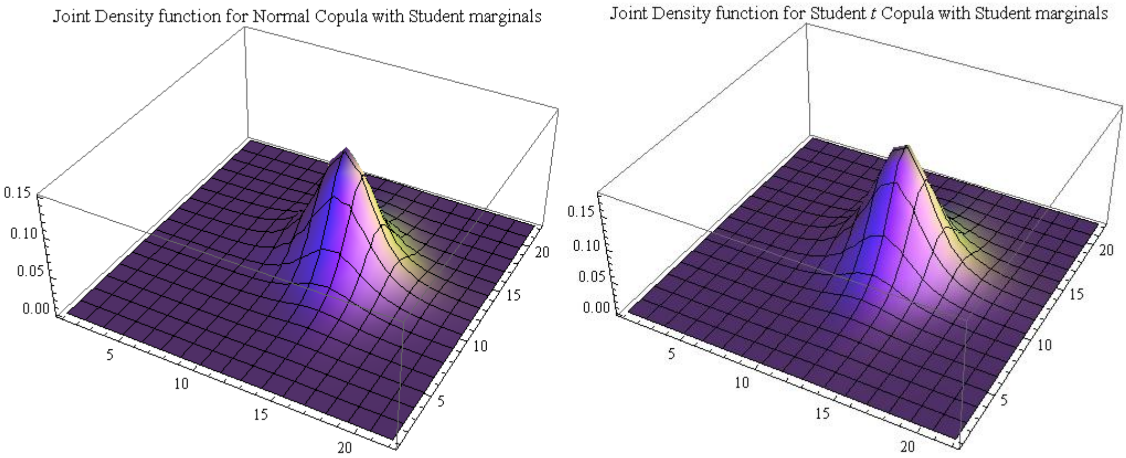
Source: Calculated by the authors with Wolfram Mathematica software on base data from finance.yahoo.com

Fig. 2: Normal copula with Student *t* marginals and Student *t* copula with student marginals



Source: Calculated by the authors with Wolfram Mathematica software on base data from finance.yahoo.com

Fig. 3: Joint density functions



Source: Calculated by the authors with Wolfram Mathematica software on base data from finance.yahoo.com

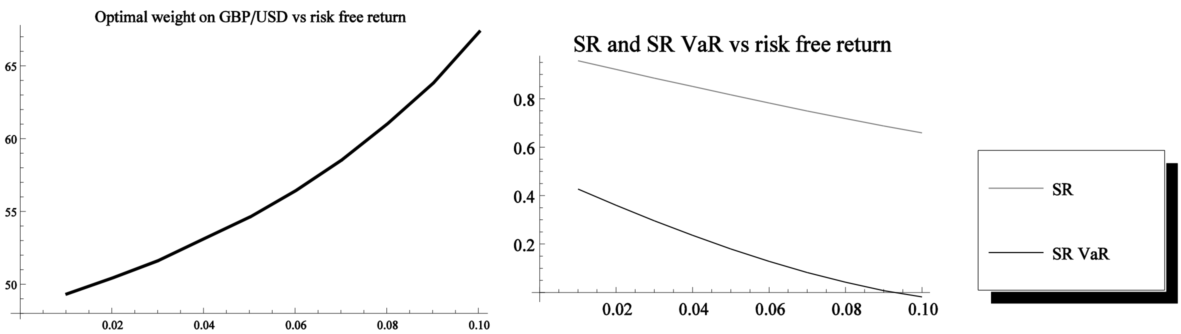
The results of the optimal portfolio weight on the GBP/USD exchange rate, the maximum Generalized Sharpe ratio based on ρ_1 (SR) and Generalized Sharpe Ratios based on ρ_2 (SR VaR) change as the risk free return ranges between 1% and 10%, when Normal copula with Student *t* marginals is used. Generalized Sharpe Ratio based on ρ_2 was computed for 10 days 5% VaR (Table 3 and Fig.4).

Table 3: Optimal weight on GBP/USD and Sharpe Ratio vs risk free return

Risk free return	0.1	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01
Weight on GBP/USD	67.4%	63.9%	61.1%	58.6%	56.5%	54.7%	53.2%	51.7%	50.5%	49.4%
Generalized Sharpe Ratio ρ_1	0.659	0.688	0.718	0.749	0.782	0.816	0.851	0.885	0.921	0.957
Generalized Sharpe Ratio ρ_2	-0.0090	0.0035	0.0201	0.0393	0.0615	0.0862	0.1132	0.1418	0.1725	0.2047

Source: Calculated by the authors using Wolfram Mathematica software on base data from finance.yahoo.com

Fig. 4: Optimal weight on GBP/USD and Sharpe Ratio vs risk free return



Source: Calculated by the authors with Wolfram Mathematica software on base data from finance.yahoo.com

The recommended amount of capital which should be invested in the GBP/USD is 53.2% and 46.8% in the AUX/USD, when risk free rate is 4%. Sharpe ratio is 0.851, Sharpe Ratio based 10 days 5% VaR is equal to 0.1132 (see Table 3). This VaR is intended to detect downward deviations of the return with respect to its expectation. VaR is usually non–negative but for heavily right skewed distributions with a “long and thin” left tail can be negative.

If we use Student *t* copula with Student marginal, case (b), for all consider risk free returns we obtain weight on GBP/USD equal to zero. It means that we recommend making an investment only into AUX/USD.

Conclusion

This paper illustrates how to select the ‘best’ allocation of the assets. We have created the portfolio that produces a portfolio returns distribution that has the best performance metric, e.g. the highest Sharpe ratio. We have used empirical returns joint distributions because then we are not limited to the multivariate normality assumption of the standard mean–variance analysis. Using an empirical distribution, all the characteristics of the joint distribution of returns on risky assets can influence the optimal allocation, not just the asset volatilities and correlations. However, a problem arises when no parametric form of joint distribution is fitted to the historical data because the optimization can produce very unstable allocations over the time.

We have shown how copulas provide a very flexible tool for modelling this joint distribution. We do not need to assume that asset returns are multivariate normal, or even elliptical, to derive optimal allocations. Generalized Sharpe ratio based on VaR tells us about risky of the composed portfolio. Their negative value means that portfolio will be more risky.

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