HANSEN-JAGANNATHAN BOUNDS AND EQUITY PREMIUM PUZZLE: A CASE OF THE CZECH CAPITAL MARKET

Vít Pošta

Abstract

The relationship between asset prices and real factors is an important part of modern economic theory and recent developments in world financial and real markets have attested its significance. The paper introduces the concept of equity premium puzzle within a stochastic discount factor model and then it presents Hansen-Jagannathan bounds as a means of both capturing this phenomena and also testing various utility function specifications, which might help to explain and solve the puzzle. The tests are run for the case of the Czech economy.

Key words: asset pricing, equity premium puzzle, Hansen-Jagannathan bounds

JEL Code: E00, E44, G12

Introduction

The paper presents a standard tool for analyzing the relationship between asset prices and macroeconomy, captured by consumption-based asset capital pricing model (CCAPM) as proposed by Lucas (1978) and Breeden (1979). It was well documented (Mehra and Prescott, 1985) that the model has trouble explaining the observed data for the US and other developed economies. The premiums measured as differences between realized equity returns and risk-free rate returns seem to be too high to be explained by the covariance between the stochastic discount factor and equity returns as the main factor. The deficiency came to be known as the equity premium puzzle, further extended in Weil (1989) by the so-called risk-free rate puzzle. The puzzle has been challanged by various theories, however, the first ones were concerned with the utility function used to describe the behavior of investors. Traditionally, CRRA utility function is preferred due to its characteristics, however, in this case the parameter of risk aversion needs to be extremely high to reconcile the model with data and even then there is the problem with the behavior of risk-free rate. Epstein and Zin (1989) proposed the so-called general expected utility function which distinguishes between the parameters of risk aversion and elasticity of intertemporal substitution and so helps to alleviate the problem of risk-free rate puzzle. Another famous modification of the utility framework is habit formation proposed by Constantinides (1990) and used in a modified version by Campbell and Cochrane (1999). The key aspect of habit formation utility functions is that utility is not derived from current consumption by itself but is based on the relation of current and past consumption. As a result economic agents have strong preference to smooth consumption and economic upturns or downturns automatically cause sufficient changes in the elasticity of intertemporal substitution without the need of setting the CRRA coeffcient at an unreasonably high level.

It is this approach to the problem of equity premium puzzle and risk-free rate puzzle which I deal with in this paper in the case of the Czech capital market. The paper is divided into three parts. In the first part I present the key results of CCAPM as a starting point to expose the problem of the two puzzles. In the second part I present the concept of Hansen-Jagannathan bounds (Hansen and Jagannathan, 1991), which will serve as the key method of empirical evaluation of the issues. In the third part I present the empirical analysis both in the form of stylized facts and also by the means of Hansen-Jagannathan bounds estimation. I summarize the main findings in the conclusion.

1 Stochastic Discount Factor

Let's assume a representative household (investor) whose preferences are described by bounded, strictly concave and increasing utility function:

$$U = E_0 \sum_{s=t}^{\infty} \beta^s u(C_s), \tag{1}$$

where β is subjective discount factor, C is real consumption, u denotes intratemporal utility function and E is expectation operator. The derivatives up to second order are assumed to be continuous. The household is constrained by:

$$A_{t+1} - A_t = Y_t + r_t A_t - C_t \quad , \tag{2}$$

where Y is real income given exogenously, r is real rate of return on asset A. The household receives income at the beginning of each period, t, and also the return on the stock of asset A. This is used to realize consumption. The difference between the total income and consumption is allocated into the asset (if negative, the asset is used to finance the excess consumption, non-negativity condition does not pose any restriction on the immediate consequences discussed below). According to (1), the household maximizes expected utility because the rate of return on the asset is assumed to be random. Solving this problem in dynamic optimization yields a standard Euler equation:

$$u'(C_t) = \beta E_t\left[(1 + r_{t+1})u'(C_{t+1})\right]. \tag{3}$$

The Euler equation asserts that the decision is optimal when marginal utility of current consumption is equal to present expected value of marginal utility of next period consumption. The other necessary condition is that the present value of the stock of asset be equal zero. Thus speculative bubbles are ruled out. The future value is discounted by the subjective discount factor which is dependent on marginal rate of time preference $\beta = \frac{1}{1+\theta}$.

The Euler equation may be expressed as:

$$1 = E_t\left[\beta \frac{u'(C_{t+1})}{u'(C_t)} R_{t+1}\right], \tag{4}$$

where R denotes gross return: $1+ r$. Alternatively, we can restate (4) as:

$$1 = E_t\left[M_{t+1}R_{t+1}\right], \tag{5}$$

where M_{t+1} is equal to $\beta \frac{u'(C_{t+1})}{u'(C_t)}$ and is called a stochastic discount factor. Using prices instead of returns, equation (5) implies:

$$p_t = E_t\left[M_{t+1}x_{t+1}\right], \tag{6}$$

where x denotes the payoff received on asset A. According to (6), current price of an asset is given by discounted value of expected future payoff. The future payoff is discounted using the stochastic discount factor. From (5) we can derive the following:

$$R^f_{t+1} = \frac{1}{E_t\left[M_{t+1}\right]}, \tag{7}$$

$$E_t R_{t+1} = R^f_{t+1} - R^f_{t+1}\,\mathrm{cov}_t\left(M_{t+1}; R_{t+1}\right). \tag{8}$$

According to (7), gross return on risk-free asset is given by an inversed value of stochastic discount factor. Equation (8) reads that expected gross return on a risky asset is given by the risk-free rate which is adjusted for the covariance between stochastic discount factor and return on the risky asset. The last term in (8) is called a risk premium.

What does stochastic discount factor exactly depend on? It is the form of utility function which gives the answer. One of the most frequently used utility functions is CRRA utility function:

$$U(C_t)=\frac{C_t^{1-\sigma}-1}{1-\sigma}. \tag{9}$$

It can be shown that the parameter σ referes to both coefficient of risk aversion (Arrow-Pratt coefficient of relative risk aversion) and elasticity of intertemporal substitution (σ being the inverse of this). Using (9), the Euler equation (9) takes on the form:

$$1=E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}R_{t+1}\right]. \tag{10}$$

According to (10) the stochastic discount factor crucially depends on the growth of consumption. Then from (8) it means that the risk premium tends to increase together with the covariance between consumption growth and returns on the risky asset. The reason is simple enough: the higher the covariance, the more difficult it is to use this asset as a hedge against economic downturns (or upturns). Consumption smoothing is then more difficult to achieve.

As implied above, Mehra and Prescott (1985) found out in the case of the US economy it is impossible for the model prediction to match the observed data when CRRA utility function is used. The risk aversion parameter needs to be calibrated at an extremely high level. If one would accept an unreasonably high σ to match the returns on the risky asset, it would lead to the model predicting an extremely high and volatile risk-free rate, known as the risk-free rate puzzle (Weil, 1989). To show this, I assume joint lognormality of consumption growth and returns and using log-approximation of the stochastic discount factor, the equations (7) and (8) may be expressed as:

$$r_{t+1}^f=\theta+\sigma E_t[\Delta c_{t+1}]-\frac{\sigma^2}{2}\operatorname{var}[\Delta c_{t+1}], \tag{11}$$

$$r_{t+1}=\theta+\sigma E_t[\Delta c_{t+1}]-\frac{\sigma^2}{2}\operatorname{var}[\Delta c_{t+1}]+\sigma\operatorname{cov}[\Delta c_{t+1},r_{t+1}], \tag{12}$$

where Δc_{t+1} denotes lnC_{t+1}-lnC_t, *var* means variance. I neglect the Jensen effect in (12). Following Cochrane (2005), the postwar real return on US capital market is estimated at 9 % (I stress the fact that it depends on the index used and it may be down to 6 %) and the real risk-free rate based on T-bills at 1 %. That means that equity premium is approaximately 8 %. From (11) and (12) it follows that equity premium should be equal to:

$$r_{t+1}-r_{t+1}^f=\sigma\operatorname{cov}[\Delta c_{t+1},r_{t+1}]. \tag{13}$$

The covariance is estimated at less than 0,2 for US economy data, therefore it requires a cofficient of relative risk aversion of more than 50, which does not seem reasonable. (A reasonable calibration of coefficient of relative risk aversion lies between 1 and 5). If accepted, (11) then implies high and volatile real risk-free rate.

Another possible utility framework is Epstein and Zin preferences. The so-called general expected utility function may be expressed as:

$$U_t = \left[(1-\delta) C_t^{1-\frac{1}{\psi}} + \delta E_t \left[U_{t+1}^{1-\sigma} \right]^{\frac{1-\frac{1}{\psi}}{1-\sigma}} \right]^{\frac{1}{1-\frac{1}{\psi}}}, \tag{14}$$

where ψ denotes elasticity of intertemporal substitution. The biggest issue is estimation of such a utility function as the expected next period utility is unobservable. On the assumption that the next period consumption (and utility) is given by the return on equity (assets), the stochastic discount factor may be expressed as (eg Smith and Wickens, 2002):

$$M_{t+1} = \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \right]^{\frac{1-\sigma}{1-\frac{1}{\psi}}} \left(R_{t+1} \right)^{\frac{1-\sigma}{1-\frac{1}{\psi}}-1}. \tag{15}$$

Finally, assuming habit formation, the utility function may be stated as:

$$U_t = \frac{(C_t - \lambda X_t)^{1-\sigma} - 1}{1-\sigma}, \tag{16}$$

where X denotes past consumption and λ is sensitivity parameter. Now assuming $\lambda = 1$ and $X_t = C_{t-1}$, the stochastic discount factor is:

$$M_{t+1} = \beta \left(\frac{C_{t+1} - C_t}{C_t - C_{t-1}} \right)^{-\sigma}. \tag{17}$$

2 Hansen-Jagannathan Bounds

In this part I will present the principles of Hansen-Jagannathan bounds and its relation to econometrics, i.e. how the estimates given in the following part were made.

Equation (5) may restated as:

$$\frac{\left| E_t R_{t+1} - R_{t+1}^f \right|}{sd\left[R_{t+1} - R_{t+1}^f \right]} = corr\left[M_{t+1}, R_{t+1} \right] \frac{sd\left[M_{t+1} \right]}{E_t M_{t+1}}, \tag{18}$$

where *sd* denotes standard deviation and *corr* stands for correlation. On the left hand side of (18) is the Sharpe ratio expressing the excess return (or equity premium) per unit of risk. Interestingly, equation (18) represents a set in which all combinations of returns and risk (standard deviations) must lie. This set is called the mean-variance set. If the correlation is 1 in absolute terms, then the set is concerned of those assets whose returns are perfectly correlated with the stochastic discount factors. Those combinations lie on the mean-variance frontier and if the correlation is -1 and risk-free rate is considered, the term capital market line coined by Sharpe (1964) is used. This in turn means that the returns lying on the frontier are also perfectly correlated among themselves and therefore can price other assets equally. Realizing that the correlation coefficient in (18) cannot be higher than 1, one can rewrite (18) as:

$$\frac{\left|E_t R_{t+1} - R_{t+1}^f\right|}{sd\left[R_{t+1} - R_{t+1}^f\right]} \leq \frac{sd\left[M_{t+1}\right]}{E_t M_{t+1}}. \tag{19}$$

The relationship given by (19) may be considered in a little different way. Given the market Sharpe ratio, a limit is set for the relation of volatility and expected value of the stochastic discount factor. This limit is independent of any utility framework and may be used to test whether or not a given utility function may be used in the analysis in the particular capital market. In other words, it tests whether or not the given stochastic discount factor based on a particular utility function may really serve as a reasonable stochastic discount factor given the capital market conditions and also the conditions of the real economy under examination. The limits given by (19) are called Hansen-Jagannathan bounds (Hansen and Jagannathan, 1991). Other restrictions on stochastic discount factors are discussed by Cochrane and Hanson (1992).

To employ this relationship it is necessary to assert a relation between the variability and observable variables (ie returns) properly; I'll loosely follow Cochrane (2005).

The idea expressed in (19) may be formulated as a projection of the stochastic discount factor on a set of returns:

$$M_t - E(M) = \left[R_t - E(R)\right]' \alpha + \varepsilon_t, \tag{20}$$

where α is a regression coefficient and ε is error assumed to be idd. The error term is not correlated with returns. Multiplying both sides of (20) by $R_t - E(R)$, expression (20) becomes:

$$E(MR) = E(M)E(R) + \Sigma\alpha, \tag{21}$$

where Σ is variance-covariance matrix of returns. Applying (5) to (21), one readily obtains:

$$\alpha = \Sigma^{-1}\left[1 - E(M)E(R)\right]. \tag{22}$$

Now expressing variance of (20):

$$\mathrm{var}(M) = \mathrm{var}\left\{[R - E(R)]' \alpha\right\} + \mathrm{var}(\varepsilon), \tag{23}$$

and using (22), one obtains an operational expression for Hansen-Jagannathan bounds:

$$\mathrm{var}(M) \geq [1 - E(M)E(R)]' \Sigma^{-1} [1 - E(M)E(R)]. \tag{24}$$

Substituing sample mean of set of returns and sample variance-covariance matrix into (24) yields a quadratic relationship between variance of stochastic discount factor and its mean. In the empirical part of the paper I use market return and risk-free rate as assets and construct Hansen-Jagannathan bounds for a set of possible means of stochastic discount factor. I estimate the given relationship using GMM.

3 Empirical Analysis

First I will present some stylized facts considering the variables used in the estimation and also reflecting on the theoretical part of the paper. Then I will present the estimates of Hansen-Jagannathan bounds and compare tham with the characteristics of stochastic discount factors under the three utility frameworks mentioned above.

3.1 Stylized Facts

I use data from OECD database. Data on real consumption, inflation, capital market return and short-term risk-free rate were retrieved to compute the variables needed for the analysis. The analysis is carried out on annual basis.

The average real return on the Czech capital market between 1995 and 2010 was app. 0,035. It is necessary to note the fact that the estimated average return on capital market may differ according to data used and due to the limited time span of the series it is susceptible to the sample chosen for an analysis. Of course, this mere fact renders the results of the analysis tentative, especially as far as the exact quantitative output is concerned.

The average real risk-free rate between 1995 and 2010 was app. 0,015. This amounts to equity premium of app. 0,02. This is relatively low compared to most advanced economies. Taking account of the variability of real return on market measured by standard deviation, which was app. 0,291, the average Sharpe ratio amounts to 0,07. This is very low compared to, for example, the US capital market, where it is estimated at app. 0,5 in the postwar data.

The average real consumption growth between 1995 and 2010 was app. 0,029 with standard deviation of 0,025.

Now assuming CRRA utility and log-normal approximation as expressed in (13), we have app. 0,02 equity premium on the left side of (13) and covariance between real market returns and real consumption growths of app. 0,0018 on the right side. This means that the parameter of risk aversion would need to amount to at least 11, which is rather high. Now using this parameter of risk aversion and substituing into (11), the real risk-free rate would amount to app. 0,297; an extremely high figure. Going from the other end; the coefficient of relative risk aversion needed for the model to fit the data on the real risk-free rate would, according to the approximate formula (11), need to be at app. 95. This by itself is an extreme value of the parameter. When substituted into (13), it would yield an equity premium of app. 0,172, clearly not the case.

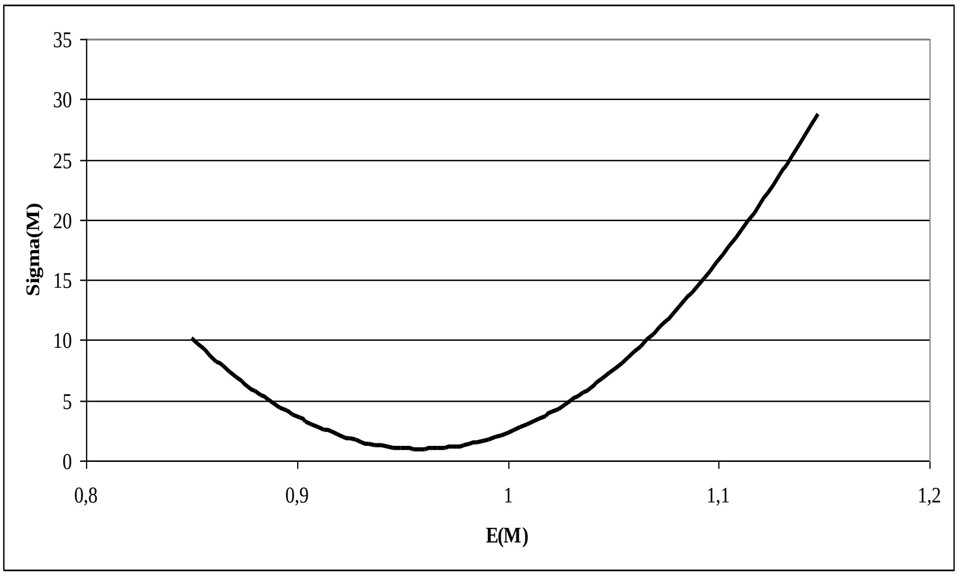
From this simple exercise, it is obvious that the CRRA utility framework cannot work to solve these issues in the case of the Czech capital market, which is a result shared among most of the economies.

To evaluate the problem more exactly, I will proceed with the estimates of Hansen-Jagannathan bounds.

3.2 Hansen-Jagannathan Bounds, CRRA Utility, Habit Formation and Epstein-Zin Preferences

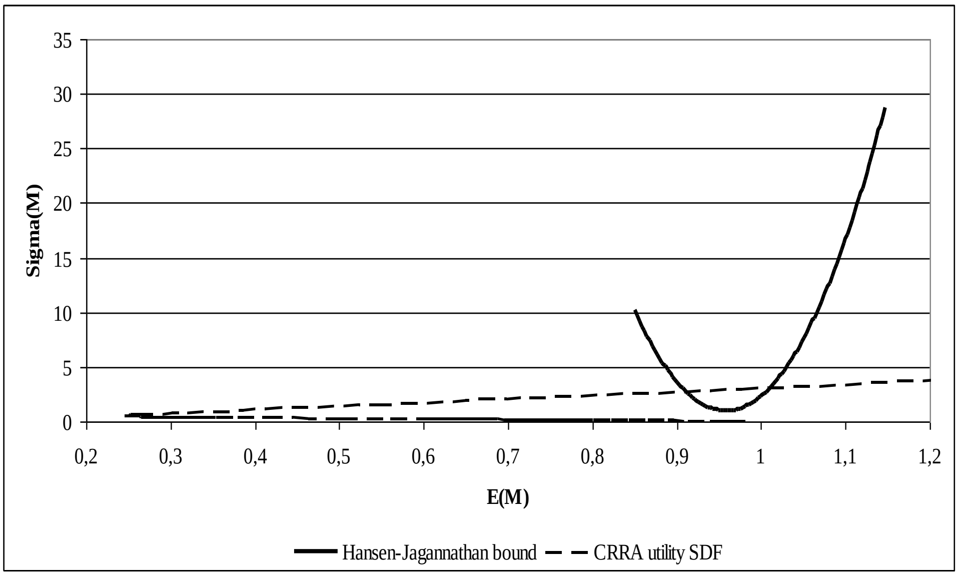
The results of the empirical analysis are presented in Figures 1 – 4. Figure 1 presents the estimate of Hansen-Jagannathan bound according to (20, 24) on an interval for mean value of stochastic discount factor (SDF) between 0,8 – 1,2. Means of stochastic discount factor above 1 is equivalent to real risk-free rate being negative. Thus, the bound defines minimum acceptable standard deviation of the stochastic discount factor given its mean value which is compatible with data on real capital market returns and real risk-free rate.

Fig. 1: Hansen-Jagannathan Bound



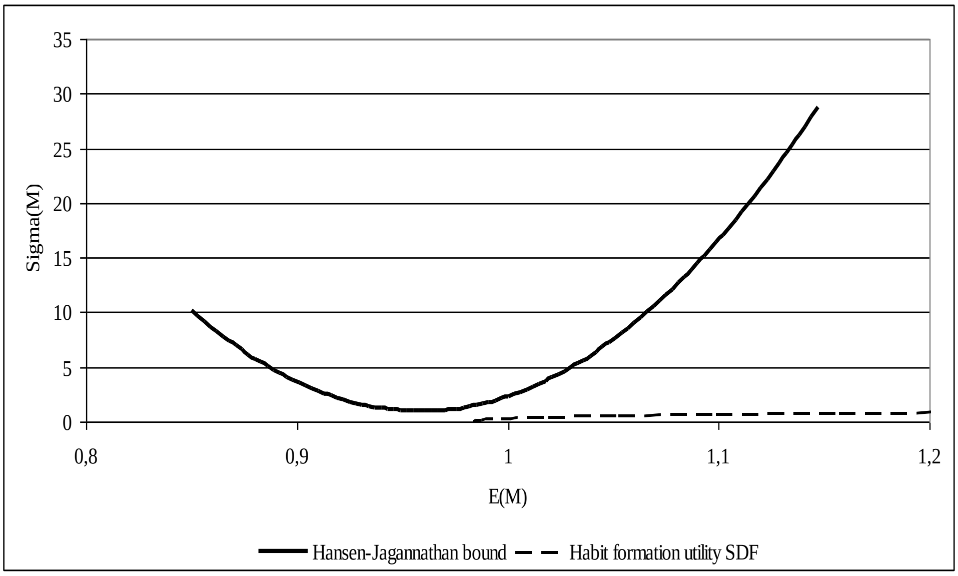
Source: My own estimates

Fig. 2: Hansen-Jagannathan Bound and SDF Based on CRRA Utility



Source: My own estimates

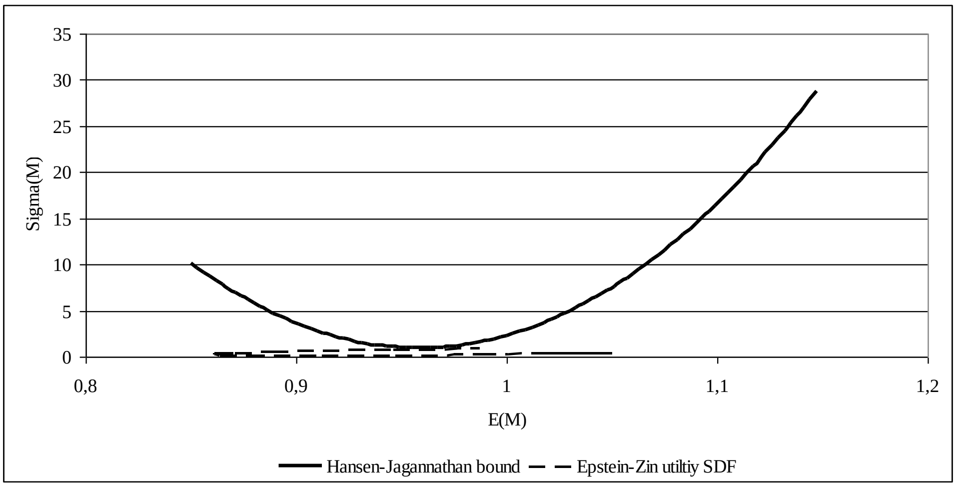
Fig. 3: Hansen-Jagannathan Bound and SDF Based on Habit Formation



Source: My own estimates

From Figure 2 it is obvious that for some values of coefficient of relative risk aversion the CRRA utility framework is able to reconcile the SDF with market data. Picking mean of SDF at 0,97 (and that is perfectly compatible with data becasuse the estimate of subjective discount factor based on real risk-free rate in the given sample is app. 0,985, in other words the real risk-free rate is app. 0,015), it has standard deviation of 2,94, which fulfills the condition posed by Hansen-Jagannathan bound as seen in Figure 2. However, coefficient of relative risk aversion of app. 700 is needed, which is out of line with economic empirics and intuition. Therefore, the CRRA utility fails to explain the equity premium. Here the estimates are not based on the log-normal approximations, but on the „raw“ formulas instead.

Fig. 4: Hansen-Jagannathan Bound and SDF Based on Epstein-Zin Utility



Source: My own estimates

In Figures 3 and 4 results with SDF based on utility function with habit formation and Espein-Zin preferences are given, respectively. Neither of these SDF solves the problem in the case of the Czech capital market. Even though they do present a little quantitative improvement, which is hardly visible in the figures, it is not enough to reconcile the real and financial data.

Conclusion

I derived key relationships concerning the real and financial variables of an economy using the consumption-based capital asset pricing model framework. The risk premium for an asset is dependent on the covariance between the asset’s returns and consumption growth, therefore is dependent on the behavior of the real economy. However, it has been shown that the CCAPM model has problem reconciling financial and real data, especially when using constant relative risk aversion utility.

One strand of approach to solving this problem rests on using different utility frameworks, especially Epstein-Zin preferences, general expected utility function, and Constantinides’s habit formation in the power utility function. These approaches are not able to solve the problem completely, but usually help to alleviate it.

I used Hansen-Jagannathan bound to test the three utility frameworks. In accordance with expectations, the CRRA utility is able to seemingly reconcile the financial and real data only for unreasonably high coefficient of relative risk aversion. However, as opposed to some research result for advanced economies, the habit formation and Epstein-Zin utility frameworks do not help much to lessen the problems which are faced with CRRA utility.

By no means, the results of the analysis are influenced by relatively short time series. However, other approaches, in my opinion especially the one of Constantinides’s relying on borrowing constraints must be considered.

References

Breeden, D. “An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities.“ *Journal of Financial Economics* 7 (3) 1979: 265-296.

Burnisede, C. “Hansen-Jagannathan Bounds as Classical Tests of Asset-Pricing Models.“ *Journal of Business & Economic Statistics* 12 (1) 1994: 57 – 79.

Campbell, J.Y., Cochrane, J.H. “By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior.“ *Journal of Political Economy* 107 (2) 1992: 205 – 251.

Cochrane, J.H., Hansen, L.P. “Asset Pricing Explorations for Macroeconomics.“ *NBER Macroeconomics Annual* 7 1992: 115 – 165.

Cochrane, J.H. *Asset Pricing*. Princeton University Press, 2005.

Constantinides, G.M. “Habit Formation: A Resolution of the Equity Premium Puzzle.“ *Journal of Political Economy* 98 (3) 1990: 519 – 543.

Epstein, L.G., Zin, S.E. “Substitution, Risk Aversion and the Temporal Behaviour of Consumption and Asset Returns: A Theoretical Approach.“ *Econometrica* 57 (4) 1989: 937 – 969.

Hansen, L.P., Jagannathan, R. “Implications of Security Market Data for Models of Dynamic Economies.“ *Journal of Political Economy* 99 (2) 1991: 225 – 262.

Lucas, R.E. “Asset Prices in an Exchange Economy.“ *Econometrica* 46 (6) 1978: 1429 – 445.

Mehra, R., Prescott, E.C. “The Equity Premium: A Puzzle.“ *Journal of Monetary Economics* 15 (2) 1985: 145 – 161.

Sharpe, W. “Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk.“ *Journal of Finance* 19 (3) 1964: 425 – 442.

Smith, P.N., Wickens, M.R. “Asset Pricing with Observable Stochastic Discount Factors.“ *Discussion Papers in Economics no 2002/03*. University of York, 2002.

Weil, P. “The Equity Premium Puzzle and the Risk-Free Rate Puzzle.“ *Journal of Monetary Economics* 24 (3) 1989: 401 – 422.

Contact

Vít Pošta

University of Economics, Prague

nám. W. Churchilla 4, Praha, Czech Republic

postav@vse.cz