STOCHASTIC MODELS OF THE TECHNOLOGICAL IDEAS FLOW AS A FOUNDATION OF THE PRODUCTION FUNCTION

Andrei Matveenko

Abstract

Only in recent decades attempts began to construct foundations of the production functions. It is natural to assume that technological processes have appeared as a result of research. In changing of production function only those technological processes are of importance which are better than their predecessors, so one can use a special branch of Probability Theory – Statistics of Extremes.

In the ideas model proposed by Jones (2005) ideas are coefficients of labor efficiency and capital efficiency of a Leontieff production function. The coefficients are drawn from Pareto distribution, moreover, the distributions describing the efficiencies of labor and capital are independent. Jones receives in asymptotics a Cobb-Douglas production function. A non-asymptotic result is received under an assumption that arrival of ideas is drawn from Poisson distribution.

In the present paper a non-asymptotic result is received without any assumptions of poissonity. Also it is demonstrated that the type of production function depends on probability distribution in the ideas model. Assuming that in the Jones‘ ideas model efficiencies of labor and capital are drawn from exponential distribution we come to a different type of production function – the CES function.

Key words: production function, flow of ideas, stochastic model, Pareto distribution, exponential distribution.

JEL code: C46, D24, E27.

Introduction

Production function is one of the basic concepts of economics. Production function

$$Y = F(x)$$

shows the output of product Y by a firm or by a country in dependence on the input of production factors $x = (x_1,...,x_n)$ (e.g. capital, labor, materials, energy, human capital, etc.) In spite of the fact that production functions since post-war time are being widely used in economics research, only in recent decades attempts began to construct foundations of the production functions, i.e. models which output are production functions.

If one associates a set of techological processes with production function then it is natural to assume that these processes have appeared as a result of research. In changing of production function only those technological processes play a role which are better than their predecessors, i.e. are *records* in some sense. Describing this kind of records one can use a special branch of Probability Theory – Statistics of Extremes (Gumbel, 1962, Beirlant et al., 2004).

Apparently, the first stochatic model of applied research was introduced by Evenson and Kislev (1976). They described a search process, consisting of n attempts leading to technologies with random productivities X_i drawn from the binomial distribution. The highest productivity was the result of the search process.

Aghion and Howitt (1992) considered a model of growth through creative destruction with sectors of intermediate goods, final good and R&D. In the simplest version of the model the output of the final good was described by a neoclassical production function $Y = \tilde{A}_k F(x)$ where x is the volume of the intermediate good. New generations k of the intermediate good appear which increase the productivity of the final good output:

$$A_{k+1} = \gamma A_k, \quad \gamma > 1.$$

The parameter γ shows the size of the innovation, the index k is the number of the innovation. The moments of innovation appearances are random. The appearance of the innovations is described by the Poisson process: during the time $[0,T]$ in average $\bar{\lambda} = \lambda n$ innovations are made, where λ is the productivity of the R&D sector, n is the qualified labor employed in the R&D sector.

Kortum (1997) introduced a model of a flow of ideas which allow, if they are better than their predecessors, to raise labor productivity. The ideas flow was described by the Poisson process.

The Kortum's model was developed by Eaton and Kortum (e.g. 1999, 2002), Alvarez, Buera and Lucas (2008), and Lucas (2009).

Jones (2005) uses the fact that each production function can be interpreted as a result of a choice of a Leontief technology from the corresponding technological menu. In the Jones' ideas model ideas are coefficients of labor efficiency and capital efficiency of a Leontief production function. The coefficients are drawn from the Pareto distribution, moreover, the distributions describing the efficiencies of labor and capital are independent. After transformations Jones receives in asymptotics a Cobb-Douglas production function. A non-asymptotic result is received under an assumption that arrival of ideas is drawn from the Poisson distribution.

In the present paper a non-asymptotic result is received without any assumptions of poissonity. Also it is demonstrated that the type of production function depends on probability distribution in the ideas model. Assuming that in the Jones' ideas model efficiencies of labor and capital are drawn from the exponential distribution we come to a different type of production function – the CES function.

1 Production function and technological menus

1.1 Basic types of production functions and relation between them

The following types of production functions are the most popular.

1) Leontief function.

$$F(x) = \min_{i=1,\dots,n} (A_i x_i)$$

Here $A_1,\dots,A_n$ are called technological coefficients or factor efficiencies.

2) Cobb-Douglas function.

$$F(x) = Ax_1^{a_1} \dots x_n^{a_n},\ 0 < a_i < 1,\ a_1 + \dots + a_n = 1$$

Here *A* is the total factor productivity (TFP).

3) CES-function.

$$F(x) = \left((A_1 x_1)^p + \dots + (A_n x_n)^p\right)^{\frac{1}{p}},\ -\infty < p < 1,\ p \neq 0$$

Under $p \to \infty$ CES-function turns into Leontief function and under $p \to 0$ CES-function turns into Cobb-Douglas function.

1.2 Representations of production functions

Let us consider production functions of the type $AF(K,L)$ where *A* is TFP, *K* is capital, *L* is labor. It is assumed that the function *F*(.,.) possesses the standard neoclassical properties (constant return to scale, increasing, decreasing returns). Well-known is the representation of the production function by use of the Euler theorem:

$$AF(K,L) = A\left(\frac{\partial F}{\partial K}K + \frac{\partial F}{\partial L}L\right).$$

Elasticities $\theta_K = \frac{\partial F}{\partial K}\frac{K}{F}$, $\theta_L = \frac{\partial F}{\partial L}\frac{L}{F}$ (where $0 < \theta_K, \theta_L < 1$; $\theta_K + \theta_L = 1$) are correspondingly *capital share* and *labor share* in the income.

Another representation of the production function is given by Jones (2005):

$$AF(K,L) = \max_{(l_K, l_L)\in\Lambda} \min\{l_K K, l_L L\}. \quad (1)$$

Here $\min\{l_K K, l_L L\}$ is Leontief production function.

Representation (1) is interpreted in the following way. A firm (or a contry) has available a set of Leontief technologies – a *technological menu* Λ. Possessing given production factors *K*, *L*, the firm (the country) chooses a Leontief technology (l_K, l_L) from the technological menu Λ search for the maximum output. In result, by the technological menu Λ, the production function standing in the left hand side of the equation (1) is received.

Jones (2005) found the technological menu

$$\Lambda = \{l_K, l_L : l_K^{\alpha} l_L^{1-\alpha} = A\},$$

leading to the Cobb-Douglas production function $AF(K,L) = AK^{\alpha}L^{1-\alpha}$ (where $0 < \alpha < 1$). Generalizing this formula and following mathematical construction of the conjugate function (Rubinov, Glover, 1998) let us define for any neoclassical production function $AF(K,L)$ a *technological menu* as the following set:

$$\Lambda = \{(l_K, l_L):\ AF\left(\frac{1}{l_K}, \frac{1}{l_L}\right) = 1\}. \quad (2)$$

THEOREM 1. *The technological menu (2) consists of all possible pairs of average products of capital and labor available under use of the production function* $AF(K,L)$.

For a concrete bundle of factors $(\tilde{K},\tilde{L})$ *the maximum in (1)* $\max_{(l_K,l_L)\in\Lambda}\min\{l_K\tilde{K},\, l_L\tilde{L}\}$ *is reached in the point of average products of capital and labor*

$$\tilde{l}_K = \frac{AF(\tilde{K},\tilde{L})}{\tilde{K}},\ \tilde{l}_L = \frac{AF(\tilde{K},\tilde{L})}{\tilde{L}}.$$

Moreover, equation (1) is fulfilled.

Proof. Equation $AF(K,L)=Y$ takes place, where Y is output. It is equivalent to equation $AF\left(\frac{K}{Y},\frac{L}{Y}\right)=1$.Hence, any admissible pair of average products of capital and labor satisfies equation $AF\left(\frac{1}{\frac{Y}{K}},\frac{1}{\frac{Y}{L}}\right)=1$, i.e. the set of such pairs enters the technological menu (2). Conversely, let us take an arbitrary point (l_K,l_L) from the technological menu. For an arbitrary Y lay $K=\frac{Y}{l_K}$, $L=\frac{Y}{l_L}$. It follows from (2) that $AF\left(\frac{1}{\frac{Y}{K}},\frac{1}{\frac{Y}{L}}\right)=1$, hence $AF(K,L)=Y$. The first part of the theorem is prooved.

To prove the second part of the theorem one needs to show that for any pair $(l_K,l_L)\in\Lambda$ the following inequality is fulfilled:

$$\min\{l_K\tilde{K},\, l_L\tilde{L}\} \le \min\{\tilde{l}_K\tilde{K},\, \tilde{l}_L\tilde{L}\} = AF(\tilde{K},\tilde{L}). \quad (3)$$

Assume the opposite: there exists such $(l_K,l_L)\in\Lambda$ that

$$l_K\tilde{K} > AF(\tilde{K},\tilde{L}),\ l_L\tilde{L} > AF(\tilde{K},\tilde{L}) \qquad (4)$$

According to the first part of the theorem there exists such point (K,L) that

$$l_K = \frac{AF(K,L)}{K},\, l_L = \frac{AF(K,L)}{L}.$$

Inequalities (4) take form

$$\frac{AF(K,L)}{K}\tilde{K} > AF(\tilde{K},\tilde{L}),\ \frac{AF(K,L)}{L}\tilde{L} > AF(\tilde{K},\tilde{L}),$$

what is equivalent to

$$AF\left(\tilde{K},\frac{L\tilde{K}}{K}\right) > AF(\tilde{K},\tilde{L}),\ AF\left(\frac{K\tilde{L}}{L},\tilde{L}\right) > AF(\tilde{K},\tilde{L}),$$

what, in its turn, is equivalent to

$$\frac{L\tilde{K}}{K} > \tilde{L},\ \frac{K\tilde{L}}{L} > \tilde{K}.$$

The latter system is incompatible what proves validity of (3). Q.E.D.

2 Models of arrival of technological ideas

2.1 Probability distributions used in the idea models

Usually in ideas models and models which are received from them after transformations one uses the following probability distributions.

1) Exponential distribution.

$$P(\xi \le x) = F(x) = 1 - e^{-\lambda x},\ \lambda > 0$$

2) Pareto distribution.

$$P(\xi \le x) = F(x) = 1 - \left(\frac{x_m}{x}\right)^{\alpha},\ x \ge x_m,\ x_m, \alpha > 0$$

3) Frechet distribution.

$$P(\xi \le x) = F(x) = e^{-\left(\frac{x-m}{s}\right)^{\alpha}},\ x > m,\ \alpha, s > 0$$

4) Poisson distribution.

$$P(\xi = k) = \frac{\lambda^k}{k!}e^{-\lambda}$$

2.2 Jones' technological ideas model

Jones (2005) for the case with two production factors (capital *K* and labor *L*) proposed an ideas model in which an idea *i* consists in a random appearance of technological coefficients a_i and b_i (efficiencies of labor and capital, correspondingly). Appearance of these two coefficients as an idea is described by independent Pareto distributions:

$$P\{a_i \le a\} = 1 - \left(\frac{a}{\gamma_a}\right)^{-\alpha},\ P\{b_i \le b\} = 1 - \left(\frac{b}{\gamma_b}\right)^{-\beta},$$

where $a \ge \gamma_a > 0,\ b \ge \gamma_b > 0,\ \alpha > 0,\ \beta > 0,\ \alpha + \beta > 1$. As far as the distributions are independent the following inequality takes place:

$$P\{b_i > b,\ a_i > a\} = \left(\frac{b}{\gamma_b}\right)^{-\beta}\left(\frac{a}{\gamma_a}\right)^{-\alpha}.$$

Under Leontief technology *i* (see the formula of Leontief production function above) the output will be higher than $\tilde{y}$ if inequalities $a_i L > \tilde{y},\ b_i K > \tilde{y}$ both take place. The probability of this event is equal to

$$P\{Y_i > \tilde{y}\} = P(a_i L > \tilde{y},\ b_i K > \tilde{y}) = \left(\frac{\tilde{y}}{K\gamma_b}\right)^{-\beta}\left(\frac{\tilde{y}}{L\gamma_a}\right)^{-\alpha}.$$

The probability of the event that output will not be higher than $\tilde{y}$ is equal to

$$P\{Y \le \tilde{y}\} = 1 - \left(\frac{\tilde{y}}{K\gamma_b}\right)^{-\beta}\left(\frac{\tilde{y}}{L\gamma_a}\right)^{-\alpha}.$$

If there arrived *N* independent ideas then the probability of the event that output is not higher than $\tilde{y}$ is equal to

$$P\{Y \le \tilde{y}\} = \left(1 - \left(\frac{\tilde{y}}{K\gamma_b}\right)^{-\beta}\left(\frac{\tilde{y}}{L\gamma_a}\right)^{-\alpha}\right)^N = \left(1 - \frac{K^\beta L^\alpha \gamma_b^\beta \gamma_a^\alpha}{\tilde{y}^{\alpha+\beta}}\right)^N.$$

Further the normalization is used:

$$z_N = K^{\frac{\beta}{\alpha+\beta}} L^{\frac{\alpha}{\alpha+\beta}} \gamma_b^{\frac{\beta}{\alpha+\beta}} \gamma_a^{\frac{\alpha}{\alpha+\beta}} N^{\frac{1}{\alpha+\beta}}.$$

Then

$$P(Y \le z_N \tilde{y}) = \left(1 - \frac{1}{N\tilde{y}^{\alpha+\beta}}\right)^N,$$

and

$$P(Y \le z_N \tilde{y}) \to e^{-\tilde{y}^{-(\alpha+\beta)}}$$

Thus: under high N the ratio Y/z_N is close to a random variable drawn from the Frechet distribution, i.e.

$$Y \approx z_N \varepsilon ,$$

where ε is drawn from the Frechet distribution.

It means that output *Y* is described by Cobb-Douglas production function with TFP coefficient which depends on the number of ideas *N* and has a random component ε drawn from the Frechet distribution.

2.3 Technological ideas model in case of exponential distributions

Now we will assume that idea *i* consisting in a random appearance of Leontief technological coefficients a_i, b_i, (labor efficiency and capital efficiency, correspondingly) is described by exponential distributions:

$$P\{a_i \le a\} = 1 - e^{-\lambda a},\ P\{b_i \le b\} = 1 - e^{-\lambda b},$$

where $a > 0,\ b > 0,\ \lambda > 0.$ As far as the distributions are independent the following inequality takes place:

$$P\{b_i > b,\ a_i > a\} = e^{-\lambda a} e^{-\lambda b}.$$

The probability of the event that under using idea *i* output is higher than $\tilde{y}$ is equal to

$$P\{Y_i > \tilde{y}\} = P(a_i L > \tilde{y},\ b_i K > \tilde{y}) = e^{-\lambda \frac{\tilde{y}}{L}} e^{-\lambda \frac{\tilde{y}}{K}} = e^{-\lambda \tilde{y}\left(\frac{1}{L} + \frac{1}{K}\right)}.$$

The probability of the event that output will not be higher than $\tilde{y}$ is equal to

$$F(\tilde{y}) = P\{Y_i \le \tilde{y}\} = 1 - e^{-\lambda \tilde{y}\left(\frac{1}{L} + \frac{1}{K}\right)}.$$

The density function equals to

$$f(\tilde{y}) = \lambda \left(\frac{1}{L} + \frac{1}{K}\right) e^{-\lambda \tilde{y}\left(\frac{1}{L} + \frac{1}{K}\right)}.$$

If there arrived *N* independent ideas then the probability of the event that output is not higher than $\tilde{y}$ is equal to

$$H_N(\tilde{y}) = P\{Y \le \tilde{y}\} = F^N(\tilde{y}) = \left(1 - e^{-\lambda \tilde{y}\left(\frac{1}{L} + \frac{1}{K}\right)}\right)^N. \qquad (5)$$

The density function equals to

$$h_N(\tilde{y}) = NF^{N-1}(\tilde{y})f(\tilde{y}) = N\left(1 - e^{-\lambda\tilde{y}\left(\frac{1}{L}+\frac{1}{K}\right)}\right)^{N-1} \lambda\left(\frac{1}{L}+\frac{1}{K}\right)e^{-\lambda\tilde{y}\left(\frac{1}{L}+\frac{1}{K}\right)}.$$

In statistics of extremes (Gumbel, 1962) it is shown that under distribution function of maximal value $\Phi_N(x) = (1 - e^{-x})^N$ mathematical expectation of the maximum value is equal to $\sum_{i=1}^{N}\frac{1}{i}$ and variance of the maximal value is equal to $\sum_{i=1}^{N}\frac{1}{i^2}$ Using these results we can calculate mathematical expectation and variance of output Y in the economy:

$$E_N(Y) = \frac{1}{\lambda}\left(\frac{1}{L}+\frac{1}{K}\right)^{-1}\sum_{i-1}^{N}\frac{1}{i} = \frac{1}{\lambda}\left(\frac{1}{L}+\frac{1}{K}\right)^{-1}\left(\ln N + \gamma + O\left(\frac{1}{N}\right)\right),$$

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant; one can see that under $N \geq 65$

$$\sum_{i=1}^{N}\frac{1}{i} \approx \ln N + 0{,}58;$$

$$D_N(Y) = \left(\frac{1}{\lambda}\left(\frac{1}{L}+\frac{1}{K}\right)^{-1}\right)^2 \sum_{i=1}^{N}\frac{1}{i^2}$$

Here $\sum_{i=1}^{n}\frac{1}{i^2}$ approaches $\frac{\pi^2}{6} \approx 1.6449$ with under $N \geq 101$ $\sum_{i=1}^{n}\frac{1}{i^2} \approx 1.64$.

This means that expected output in the economy is defined by CES production function with parameter $p = -1$ and with TFP coefficient depending on the number of technological ideas *N*. Moreover, standard deviation depends on arguments of production function *K* and *L* but does not depend on TFP.

Thus, under an assumption of exponential distribution of the ideas we have received a non-asymptotic result on a type of production function without any assumptions on how precisely do ideas arrive. Jones (2005), under assumption that the productivities of ideas are drawn from the Pareto distribution, had to make additional assumption that the arrival of ideas is described by the Poisson distribution.

To receive an asymptotic result let us use in connection with (5) the following normalization:

$$z_N = \left(\frac{1}{L}+\frac{1}{K}\right)^{-1}\frac{1}{\lambda}\ln N$$

then

$$P(Y \le z_N \tilde{y}) = \left(1 - \left(\frac{1}{N}\right)^{\tilde{y}}\right)^N$$

and under $N \to \infty$ convergence in probability takes place:

$$P(Y \le z_N \tilde{y}) \to \begin{cases} 0, & \tilde{y} < 1 \\ 1, & \tilde{y} \ge 1 \end{cases}.$$

Notice that the pointwise convergence

$$P(Y \le z_N \tilde{y}) \to \begin{cases} 0, & \tilde{y} < 1 \\ e^{-1}, & \tilde{y} = 1 \\ 1, & \tilde{y} > 1 \end{cases}$$

under $N \to \infty$ does not lead to a distribution function.

I.e.

$$Y = z_N$$

with probability 1.

This means that output with probability 1 is described by CES-function z_N with parameter $p = -1$ with TFP coefficient which depends on number of ideas:

$$F(K, L) = \frac{1}{\lambda} \ln N \left(\frac{1}{L} + \frac{1}{K}\right)^{-1}.$$

2.4 Application of exponential distribution as a tool for analysis of the Jones' case with Pareto distribution

Assuming that arrival of ideas is described by Poisson distribution and productivity of ideas is described by Pareto distribution. Jones (2005) received the following expected value of output in economy:

$$E(Y) = \Gamma\left(1 - \frac{1}{\alpha + \beta}\right)\left(\kappa N K^{\beta} L^{\alpha}\right)^{\frac{1}{\alpha+\beta}}.$$

Here N is the number of ideas, κ is a constant, $\Gamma(.)$ is gamma-function. Thus, Cobb-Douglas production function is received.

Let us calculate logarithm of expected output using an expansion of logarithm of gamma-function (Gumbel, 1962):

$$\ln E(Y) = \ln\left(K^{\frac{\beta}{\alpha+\beta}} L^{\frac{\alpha}{\alpha+\beta}} \kappa^{\frac{1}{\alpha+\beta}}\right) + \frac{1}{\alpha + \beta} \ln N + \gamma \frac{1}{\alpha + \beta} + \sum_{\upsilon=2}^{\infty} S_{\upsilon} \frac{\left(\frac{1}{\alpha + \beta}\right)^{\upsilon}}{\upsilon},$$

where γ is Euler-Mascheroni constant,

$$S_\upsilon = \sum_{\lambda=1}^{\infty} \lambda^{-\upsilon} .$$

Let us show that similar result can be received without invoking poissonity assumption.

$$P(Y \le \tilde{y}) = \left(1 - \frac{K^\beta L^\alpha \gamma_b^\beta \gamma_a^\alpha}{\tilde{y}^{\alpha+\beta}}\right)^N = \left(1 - e^{\ln\left(\frac{K^\beta L^\alpha \gamma_b^\beta \gamma_a^\alpha}{\tilde{y}^{\alpha+\beta}}\right)}\right)^N =$$

$$= \left(1 - e^{\left(\ln\left(K^\beta L^\alpha \gamma_b^\beta \gamma_a^\alpha\right) - \ln\left(\tilde{y}^{\alpha+\beta}\right)\right)}\right)^N = \left(1 - e^{-(\alpha+\beta)\left(\ln\tilde{y} - \ln\left(K^{\frac{\beta}{\alpha+\beta}} L^{\frac{\alpha}{\alpha+\beta}} \gamma_b^{\frac{\beta}{\alpha+\beta}} \gamma^{\frac{\alpha}{\alpha+\beta}}\right)\right)}\right)^N .$$

Hence,

$$P(\ln Y \le \ln\tilde{y}) = \left(1 - e^{-(\alpha+\beta)\left(\ln\tilde{y} - \ln\left(K^{\frac{\beta}{\alpha+\beta}} L^{\frac{\alpha}{\alpha+\beta}} \gamma_b^{\frac{\beta}{\alpha+\beta}} \gamma^{\frac{\alpha}{\alpha+\beta}}\right)\right)}\right)^N .$$

Expected value and variance of logarithm of output in economy equals:

$$E_N(\ln Y) = \ln\left(K^{\frac{\beta}{\alpha+\beta}} L^{\frac{\alpha}{\alpha+\beta}} \gamma_b^{\frac{\beta}{\alpha+\beta}} \gamma_a^{\frac{\alpha}{\alpha+\beta}}\right) + \frac{1}{\alpha+\beta}\sum_{i=1}^{N}\frac{1}{i} =$$

$$= \ln\left(K^{\frac{\beta}{\alpha+\beta}} L^{\frac{\alpha}{\alpha+\beta}} \gamma_b^{\frac{\beta}{\alpha+\beta}} \gamma_a^{\frac{\alpha}{\alpha+\beta}}\right) + \frac{1}{\alpha+\beta}\ln N + \gamma\frac{1}{\alpha+\beta} + O\left(\frac{1}{N}\right),$$

$$D_N(\ln Y) = \frac{1}{(\alpha+\beta)^2}\sum_{i=1}^{N}\frac{1}{i^2} .$$

Conclusion

Jones (2005) proposed an ideas model where coefficients were drawn from Pareto distribution. Jones received in asymptotics a Cobb-Douglas production function. A non-asymptotic result was received under an assumption that arrival of ideas is drawn from Poisson distribution.

In the present paper a non-asymptotic result is received without any assumptions of poissonity. Also it is demonstrated that the type of production function depends on probability distribution in the ideas model. Assuming that in the Jones‘ ideas model efficiencies of labor and capital are drawn from exponential distribution we come to a different type of production function – the CES function.

Acknowledgement

The research is supported by the Russian Foundation for Basic Research (project 11-01-00878a). The author is grateful to Sergei Vallander for useful comments.

References

Aghion, P. & Howitt, P. (1992) A model of growth through creative destruction *Econometrica*, *60*(2), 323–351.

Gumbel, E. J. (1962) *The Statistics of Extremes.* New York: Columbia University Press.

Alvarez, F., Buera, F. & Lucas, R. (2008) Models of idea flows. NBER Working Paper Series, 14135.

Beirlant, J., Goegebeur, Y., Teugels, J. & Segers, J. (2004) *Statistics of Extremes: Theory and Applications*. Chichester: John Willey & Sons.

Eaton, J. & Kortum, S. (1999) International technology diffusion: theory and measurement. *International Economic Review*, *40*(3), 537–570.

Eaton, J. & Kortum, S. (2002) Technology, geography and trade. *Econometrica*, *70*(5), 1741–1779.

Evenson, R.E. & Kislev, Y. (1976) A stochastic model of applied research. *Journal of Political Economy*, *84*(2), 265–282.

Jones, C.I. (2005) The shape of production function and the direction of technical change. *Quarterly Journal of Economics*, *120*(2), 517–549.

Kortum, S. (1997) Research, patenting and technological change. *Econometrica*, *65*(6), 1389–1419.

Lucas, R. (2009) Ideas and growth. *Economica*, *76*(301), 1–19.

Contact

Andrei Matveenko

Department of Probability Theory and Statistics, Faculty of Mathematics and Mechanics, St. Petersburg State University

7-9, Universitetskaya nab., St. Petersburg, Russia

andrei.v.matve@gmail.com