GEOMETRY ONE SPECIAL TYPE OF SURFACES IN R[3]

Miloš Kaňka - Eva Kaňková

Abstract

The aim of this paper is to give formulas for Gaussian and Mean curvature of one special type of surfaces of the form

$$x_1^\alpha + x_2^\alpha + x_3^\alpha = 1, \quad \text{where} \quad \alpha \in R, \quad \alpha > 0, \quad \alpha \neq 1. \tag{1}$$

Key words: Special type of surfaces in R^3, tangent vectors, unit normal vector, first fundamental form, second fundamental form, Weingarten map, Gaussian curvature, Mean curvature

JEL Code: C00

Introduction

In [4] we studied the Gaussian curvature of surfaces of this type without using their parametrical description. To reach the formulas of Gaussian and Mean Curvature, we use in this remark parametrical description of (1) in the form

$$x_3 = f(x_1, x_2) = (1 - x_1^\alpha - x_2^\alpha)^{1/\alpha}, \quad 1 - x_1^\alpha - x_2^\alpha > 0.$$

1 My results

The tangent and normal vectors at the arbitrary point $x = (x_1, x_2, x_3) \in S$ are:

$$\mathbf{g}_{x_1} = (1,\ 0,\ f_{x_1}(x_1, x_2)), \quad \mathbf{g}_{x_2} = (0,\ 1,\ f_{x_1}(x_1, x_2)), \quad \mathbf{n} = (-f_{x_1}(x_1, x_2),\ -f_{x_2}(x_1, x_2),\ 1).$$

In case of (1) the tangent vectors and the normal vector have the form

$$\mathbf{g}_{x_1} = \left(1,\ 0,\ \frac{-x_1^{\alpha-1}}{\left(1 - x_1^\alpha - x_2^\alpha\right)^{\frac{\alpha-1}{\alpha}}}\right), \qquad \mathbf{g}_{x_2} = \left(0,\ 1,\ \frac{-x_2^{\alpha-1}}{\left(1 - x_1^\alpha - x_2^\alpha\right)^{\frac{\alpha-1}{\alpha}}}\right), \tag{2}$$

$$\mathbf{n} = \left(\frac{x_1^{\alpha-1}}{\left(1 - x_1^\alpha - x_2^\alpha\right)^{\frac{\alpha-1}{\alpha}}},\ \frac{x_2^{\alpha-1}}{\left(1 - x_1^\alpha - x_2^\alpha\right)^{\frac{\alpha-1}{\alpha}}},\ 1\right). \tag{3}$$

It is also possible to rewrite vectors (2) and (3) in the form

$$\mathbf{g}_{x_1} = \left(1,\ 0,\ -\left(\frac{x_1}{x_3}\right)^{\alpha-1}\right), \qquad \mathbf{g}_{x_2} = \left(0,\ 1,\ -\left(\frac{x_2}{x_3}\right)^{\alpha-1}\right), \qquad \mathbf{n} = \left(\left(\frac{x_1}{x_3}\right)^{\alpha-1},\ \left(\frac{x_2}{x_3}\right)^{\alpha-1},\ 1\right).$$

The unit normal N has the form

$$N = \frac{1}{\left(x_1^{2\alpha-2} + x_2^{2\alpha-2} + x_3^{2\alpha-2}\right)^{1/2}} \cdot \left(x_1^{\alpha-1},\ x_2^{\alpha-1},\ x_3^{\alpha-1}\right).$$

The equation $N \cdot N = 1$ gives

$$(N \cdot N)_{x_1} = 0 \ \wedge \ (N \cdot N)_{x_2} = 0 \ \Rightarrow \ 2 \cdot N \cdot N_{x_1} = 0 \ \wedge \ 2 \cdot N \cdot N_{x_1} = 0 . \tag{4}$$

From (4) follows that $N_{x_1}, N_{x_2} \in T_x(S)$. We can express the vectors N_{x_1} and N_{x_2} as a linear combination of the bases of $T_x(S)$, where $x = (x_1, x_2, x_3)$. So we have

$$\begin{aligned} N_{x_1} &= a_{11} g_{x_1} + a_{12} g_{x_2}, \\ N_{x_2} &= a_{21} g_{x_1} + a_{22} g_{x_2}. \end{aligned} \tag{5}$$

From (5) follows

$$\begin{aligned} N_{x_1} \cdot g_{x_1} &= a_{11} g_{11} + a_{12} g_{12}, \\ N_{x_2} \cdot g_{x_1} &= a_{21} g_{11} + a_{22} g_{12}, \end{aligned} \quad \text{or} \quad \begin{aligned} N_{x_1} \cdot g_{x_2} &= a_{11} g_{12} + a_{12} g_{22}, \\ N_{x_2} \cdot g_{x_2} &= a_{21} g_{12} + a_{22} g_{22}, \end{aligned} \tag{6}$$

where functions

$$g_{11} = 1 + \frac{x_1^{2\alpha-2}}{\left(1 - x_1^{\alpha} - x_2^{\alpha}\right)^{\frac{2\alpha-2}{\alpha}}}, \quad g_{12} = \frac{(x_1 \cdot x_2)^{\alpha-1}}{\left(1 - x_1^{\alpha} - x_2^{\alpha}\right)^{\frac{2\alpha-2}{\alpha}}}, \quad g_{22} = 1 + \frac{x_2^{2\alpha-2}}{\left(1 - x_1^{\alpha} - x_2^{\alpha}\right)^{\frac{2\alpha-2}{\alpha}}},$$

which can be also written in the form

$$g_{11} = 1 + \left(\frac{x_1}{x_3}\right)^{2\alpha-2}, \quad g_{12} = \left(\frac{x_1}{x_3}\right)^{\alpha-1} \cdot \left(\frac{x_2}{x_3}\right)^{\alpha-1}, \quad g_{22} = 1 + \left(\frac{x_2}{x_3}\right)^{2\alpha-2},$$

are the coefficients of the first fundamental form

$$g_{11}\, dx_1^2 + 2 g_{12}\, dx_1\, dx_2 + g_{22}\, dx_2^2 .$$

On the other hand, we have

$$\begin{aligned} N \cdot g_{x_1} = 0 \ &\Rightarrow \ N_{x_1} \cdot g_{x_1} + N \cdot g_{x_1 x_1} = 0 \ \Rightarrow \ N \cdot g_{x_1 x_1} = -N_{x_1} \cdot g_{x_1}, \\ N \cdot g_{x_1} = 0 \ &\Rightarrow \ N_{x_2} \cdot g_{x_1} + N \cdot g_{x_1 x_2} = 0 \ \Rightarrow \ N \cdot g_{x_1 x_2} = -N_{x_2} \cdot g_{x_1} \end{aligned}$$

and analogically

$$N \cdot g_{x_2} = 0 \;\Rightarrow\; N_{x_2} \cdot g_{x_2} + N \cdot g_{x_2 x_2} = 0 \;\Rightarrow\; N \cdot g_{x_2 x_2} = -N_{x_2} \cdot g_{x_2},$$
$$N \cdot g_{x_1} = 0 \;\Rightarrow\; N_{x_2} \cdot g_{x_1} + N \cdot g_{x_1 x_2} = 0 \;\Rightarrow\; N \cdot g_{x_1 x_2} = -N_{x_2} \cdot g_{x_1}.$$

The functions

$$G_{11} = N \cdot g_{x_1 x_1}, \quad G_{12} = G_{21} = N \cdot g_{x_1 x_2}, \quad G_{22} = N \cdot g_{x_2 x_2}$$

are the coefficients of the second fundamental form

$$G_{11}\, dx_1^2 + 2G_{12}\, dx_1\, dx_2 + G_{22}\, dx_2^2 .$$

The functions G_{11}, G_{22}, G_{12} have the form

$$G_{11} = \frac{(\alpha - 1) \cdot x_1^{\alpha-2} \cdot (x_2^{\alpha} - 1)}{x_3^{\alpha} \cdot \left[\left(x_1^{2\alpha-2} + x_2^{2\alpha-2} + x_3^{2\alpha-2}\right)\right]^{1/2}},$$

$$G_{22} = \frac{(\alpha - 1) \cdot x_2^{\alpha-2} \cdot (x_1^{\alpha} - 1)}{x_3^{\alpha} \cdot \left[\left(x_1^{2\alpha-2} + x_2^{2\alpha-2} + x_3^{2\alpha-2}\right)\right]^{1/2}},$$

$$G_{12} = \frac{(1 - \alpha) \cdot x_1^{\alpha-1} \cdot x_2^{\alpha-1}}{x_3^{\alpha} \cdot \left[\left(x_1^{2\alpha-2} + x_2^{2\alpha-2} + x_3^{2\alpha-2}\right)\right]^{1/2}}.$$

Thanks to equations (6) we obtain

$$\begin{aligned} G_{11} &= -a_{11}g_{11} - a_{12}g_{12}, \\ G_{12} &= -a_{21}g_{11} - a_{22}g_{12}, \end{aligned} \quad \text{or} \quad \begin{aligned} G_{12} &= -a_{11}g_{12} - a_{12}g_{22}, \\ G_{22} &= -a_{21}g_{12} - a_{22}g_{22}. \end{aligned} \tag{7}$$

Equations (7) have the form

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{pmatrix} = -\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

or finally the form

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = -\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}^{-1} \cdot \begin{pmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{pmatrix},$$

which means

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \frac{-1}{g_{11} \cdot g_{22} - g_{12}^2} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix} \cdot \begin{pmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{pmatrix}. \tag{8}$$

From (8) we can obtain the real form of functions $a_{11}, a_{12}, a_{21}, a_{22}$:

$$a_{11} = \frac{G_{11} \cdot g_{22} - G_{12} \cdot g_{12}}{g_{12}^2 - g_{11} \cdot g_{22}}, \quad a_{21} = \frac{G_{12} \cdot g_{22} - G_{22} \cdot g_{12}}{g_{12}^2 - g_{11} \cdot g_{22}},$$

$$a_{12} = \frac{-G_{11} \cdot g_{12} + G_{12} \cdot g_{11}}{g_{12}^2 - g_{11} \cdot g_{22}}, \quad a_{22} = \frac{-G_{12} \cdot g_{12} + G_{22} \cdot g_{11}}{g_{12}^2 - g_{11} \cdot g_{22}}.$$

Substituting into (5) we obtain

$$-N_{x_1} = \frac{G_{11} \cdot g_{22} - G_{12} \cdot g_{12}}{g_{11} \cdot g_{22} - g_{12}^2} g_{x_1} + \frac{-G_{11} \cdot g_{12} + G_{12} \cdot g_{11}}{g_{11} \cdot g_{22} - g_{12}^2} g_{x_2},$$

$$-N_{x_2} = \frac{G_{12} \cdot g_{22} - G_{22} \cdot g_{12}}{g_{11} \cdot g_{22} - g_{12}^2} g_{x_1} + \frac{-G_{12} \cdot g_{12} + G_{22} \cdot g_{11}}{g_{11} \cdot g_{22} - g_{12}^2} g_{x_2}.$$

The Weingarten map defined for regular surfaces S by the formula

$$W(v_p) = -N_v,$$

where $v_p \in T_p(S)$ and N is a unit normal defined in a neighborhood of a point $p \in S$, and N_v is the derivative with respect to v_p. So we have

$$W(g_{x_1}) = -N_{x_1}, \quad \text{and} \quad W(g_{x_2}) = -N_{x_2}.$$

The Gaussian curvature K equals the determinant $\det W$ which means the determinant

$$K = \det \begin{pmatrix} \frac{G_{11} \cdot g_{22} - G_{12} \cdot g_{12}}{g_{11} \cdot g_{22} - g_{12}^2} & \frac{-G_{11} \cdot g_{12} + G_{12} \cdot g_{11}}{g_{11} \cdot g_{22} - g_{12}^2} \\ \frac{G_{12} \cdot g_{22} - G_{22} \cdot g_{12}}{g_{11} \cdot g_{22} - g_{12}^2} & \frac{-G_{12} \cdot g_{12} + G_{22} \cdot g_{11}}{g_{11} \cdot g_{22} - g_{12}^2} \end{pmatrix}.$$

So we have

$$K = \frac{G_{11} \cdot G_{22} - G_{12}^2}{g_{11} \cdot g_{22} - g_{12}^2}.$$

The mean curvature equals the trace of the matrix

$$H = \frac{1}{2} tr \begin{pmatrix} \frac{G_{11} \cdot g_{22} - G_{12} \cdot g_{12}}{g_{11} \cdot g_{22} - g_{12}^2} & \frac{-G_{11} \cdot g_{12} + G_{12} \cdot g_{11}}{g_{11} \cdot g_{22} - g_{12}^2} \\ \frac{G_{12} \cdot g_{22} - G_{22} \cdot g_{12}}{g_{11} \cdot g_{22} - g_{12}^2} & \frac{-G_{12} \cdot g_{12} + G_{22} \cdot g_{11}}{g_{11} \cdot g_{22} - g_{12}^2} \end{pmatrix}.$$

The detailed calculation gives

$$K = \frac{\frac{(\alpha-1)^2 x_1^{\alpha-2} x_2^{\alpha-2} (x_1^\alpha - 1)(x_2^\alpha - 1) - (\alpha-1)^2 x_1^{2\alpha-2} x_2^{2\alpha-2}}{x_3^{2\alpha}[x_1^{2\alpha-2} + x_2^{2\alpha-2} + x_3^{2\alpha-2}]}}{\frac{[x_1^{2\alpha-2} + x_2^{2\alpha-2} + x_3^{2\alpha-2}]}{x_3^{2\alpha-2}}} =$$

$$= \frac{\frac{(\alpha-1)^2 x_1^{\alpha-2} x_2^{\alpha-2} [1 - x_1^\alpha - x_2^\alpha + x_1^\alpha x_2^\alpha - x_1^\alpha x_2^\alpha]}{x_3^{2\alpha}[x_1^{2\alpha-2} + x_2^{2\alpha-2} + x_3^{2\alpha-2}]}}{\frac{[x_1^{2\alpha-2} + x_2^{2\alpha-2} + x_3^{2\alpha-2}]}{x_3^{2\alpha-2}}} =$$

$$= \frac{(\alpha-1)^2 x_3^{-2} x_1^{\alpha-2} x_2^{\alpha-2} x_3^\alpha}{[x_1^{2\alpha-2} + x_2^{2\alpha-2} + x_3^{2\alpha-2}]^2} = \frac{(\alpha-1)(x_1 x_2 x_3)^{\alpha-2}}{(x_1^{2\alpha-2} + x_2^{2\alpha-2} + x_3^{2\alpha-2})^2}.$$

Further we have

$$H = \frac{G_{11} \cdot g_{22} - 2 G_{12} \cdot g_{12} + G_{22} \cdot g_{11}}{2(g_{11} \cdot g_{22} - g_{12}^2)}.$$

In case of special surfaces (1) we obtain

$$H = \frac{\left(1 + \frac{x_2^{2\alpha-2}}{x_3^{2\alpha-2}}\right) \frac{(1-\alpha) x_1^{\alpha-2} (1 - x_2^\alpha)}{x_3^{2\alpha-1}} + 2 \frac{(\alpha-1) x_1^{2\alpha-2} x_2^{2\alpha-2}}{x_3^{4\alpha-3}}}{2\left(\frac{x_1^{2\alpha-2} + x_2^{2\alpha-2} + x_3^{2\alpha-2}}{x_3^{2\alpha-2}}\right)^{3/2}} + \frac{\left(1 + \frac{x_1^{2\alpha-2}}{x_3^{2\alpha-2}}\right) \frac{(1-\alpha) x_2^{\alpha-2} (1 - x_1^\alpha)}{x_3^{2\alpha-1}}}{2\left(\frac{x_1^{2\alpha-2} + x_2^{2\alpha-2} + x_3^{2\alpha-2}}{x_3^{2\alpha-2}}\right)^{3/2}}.$$

The formula for H is

$$H = \frac{(1-\alpha) \frac{1}{x_3^\alpha} A}{2[x_1^{2\alpha-2} + x_2^{2\alpha-2} + x_3^{2\alpha-2}]^{3/2}},$$

where

$$A = (x_3^{2\alpha-2} + x_2^{2\alpha-2}) x_1^{\alpha-2} (1 - x_2^\alpha) - 2 x_1^{2\alpha-2} x_2^{2\alpha-2} + (x_3^{2\alpha-2} + x_1^{2\alpha-2}) x_2^{\alpha-2} (1 - x_1^\alpha) =$$
$$= x_3^{2\alpha-2} x_1^{\alpha-2} (x_1^\alpha + x_3^\alpha) + x_2^{2\alpha-2} x_1^{\alpha-2} (1 - x_2^\alpha - x_1^\alpha) + x_1^{2\alpha-2} x_2^{\alpha-2} (1 - x_1^\alpha - x_2^\alpha) + x_3^{2\alpha-2} x_2^{\alpha-2} (1 - x_1^\alpha).$$

We have

$$(1-\alpha)\frac{1}{x_3^{\alpha}}A=(1-\alpha)[x_3^{\alpha-2}x_1^{\alpha-2}(x_1^{\alpha}+x_3^{\alpha})+x_1^{\alpha-2}x_2^{\alpha-2}(x_2^{\alpha}+x_1^{\alpha})+x_2^{\alpha-2}x_3^{\alpha-2}(x_2^{\alpha}+x_3^{\alpha})].$$

The formula for Mean curvature can be written in the form

$$H=\frac{(1-\alpha)\left[(x_1x_2)^{\alpha-2}(x_1^{\alpha}+x_2^{\alpha})+(x_2x_3)^{\alpha-2}(x_2^{\alpha}+x_3^{\alpha})+(x_1x_3)^{\alpha-2}(x_1^{\alpha}+x_3^{\alpha})\right]}{2\left(x_1^{2\alpha-2}+x_2^{2\alpha-2}+x_3^{2\alpha-2}\right)^{3/2}}.$$

Conclusion

Gaussian and mean curvatures of special surface (1) are

$$K=\frac{(\alpha-1)(x_1x_2x_3)^{\alpha-2}}{(x_1^{2\alpha-2}+x_2^{2\alpha-2}+x_3^{2\alpha-2})^2}$$

and

$$H=\frac{(1-\alpha)\left[(x_1x_2)^{\alpha-2}(x_1^{\alpha}+x_2^{\alpha})+(x_2x_3)^{\alpha-2}(x_2^{\alpha}+x_3^{\alpha})+(x_1x_3)^{\alpha-2}(x_1^{\alpha}+x_3^{\alpha})\right]}{2\left(x_1^{2\alpha-2}+x_2^{2\alpha-2}+x_3^{2\alpha-2}\right)^{3/2}}.$$

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Contact

Miloš Kaňka

University of Economics, Faculty of Informatics and Statistics, Department of Mathematics,

Winston Churchill square 4, Prague 3, Czech Republic

kanka@vse.cz

Eva Kaňková

University of Economics, Faculty of Business Administration, Department of Microeconomics, Winston Churchill square 4, Prague 3, Czech Republic

kankova@vse.cz