SOME FUNCTIONS IN ECONOMY FROM MATHEMATICAL POINT OF VIEW

(Application of Cartan's moving frame method.)

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Abstract

The aim of this article is to give basic geometrical characteristic of some utility functions used in economics. We are going to study these functions as regular surfaces in R^3. Applying the method of Cartan moving frame we obtain geometrical description of production function

$$f(u,v) = A \cdot u^{\alpha} \cdot v^{\beta}, \text{ where } A = 1,\ \alpha = 1 \text{ or } \alpha = 2,\ \beta = 1,$$

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Introduction

Let us suppose that for every point $p \in M \subset R^3$ exist an open set $U \subset R^2$, an open set $V \subset R^3$, $p \in V$, and a homeomorphism $x: U \to V \cap M$. A subset $M \subset R^3$ is called a two-dimensional surface in R^3. Let $x(U) \subset V \cap M \subset R^3$ be a neighbourhood of $p \in M$ such that the restriction $x \,|\, U$ is an embedding into $x(U) = V \cap M$ and that it is possible to choose in $x(U)$ an orthonormal moving frame $\{E_1, E_2, E_3\}$ in such a way that E_1, E_2 are tangent to $x(U)$ and E_3 is a non-vanishing normal to $x(U)$. We first discuss the Cartan structural equations for a two-dimensional surface in R^3.

1. Basic equations

Differentiating a map $x(u,v)$ we obtain

$$dx = x_u \, du + x_v \, dv,$$

where x_u , x_v are tangent vector fields. Let us denote

$$n(u,v) = \frac{x_u \times x_v}{|x_u \times x_v|}$$

the unit normal vector field. With respect to the orthonormal moving frame $\{E_1, E_2, E_3\}$ we have

$$dx = E_1\theta_1 + E_2\theta_2 + E_3\theta_3.$$

where $\theta_i(E_j) = \delta_{ij}$. Since x_u and x_v are tangent to $x(U)$ we have $dx \cdot E_3 = 0$ which implies $\theta_3 = 0$ and

$$dx = E_1\theta_1 + E_2\theta_2.$$

Each vector $E_i : U \subset R^3 \to R^3$ is a differentiable map and the differential

$$dE_i : R^3 \to R^3$$

is a linear map. So we may write (using Einstein's notation)

$$dE_i = \omega_{ij} E_{j.}$$

where ω_{ij} are linear forms on R^3 and since E_i are differentiable ω_{ij} are nine differentiable forms. So we have

$$\begin{aligned} dE_1 &= \omega_{11}E_1 + \omega_{12}E_2 + \omega_{13}E_{3.} \\ dE_1 &= \omega_{21}E_1 + \omega_{22}E_2 + \omega_{23}E_{3.} \\ dE_3 &= \omega_{31}E_1 + \omega_{32}E_2 + \omega_{33}E_{3.} \end{aligned} \tag{1}$$

Differentiating equation $E_i \cdot E_j = \delta_{ij}$ we obtain

$$dE_i E_j + E_i dE_j = \omega_{ij} + \omega_{ji} = 0.$$

Forms ω_{ij} are antisymmetric

$$\omega_{ii} = 0 \quad , \quad \omega_{ij} = -\omega_{ji} \tag{2}$$

From (1) and (2) we have

$$\begin{aligned} dE_1 &= \qquad\qquad \omega_{12}E_2 + \omega_{13}E_3, \\ dE_2 &= -\omega_{12}E_1 \qquad\qquad + \omega_{23}E_3, \\ dE_3 &= -\omega_{13}E_1 - \omega_{23}E_2. \end{aligned} \tag{3}$$

Forms dx and dE_i have vanishing exterior derivatives

$$0 = d^2x = dE_1 \wedge \theta_1 + E_1 d\theta_1 + dE_2 \wedge \theta_2 + E_2 d\theta_2. \tag{4}$$

Substituting (3) into (4) we obtain

$$(\omega_{12}E_2 + \omega_{13}E_3) \wedge \theta_1 + E_1 d\theta_1 + (\omega_{21}E_1 + \omega_{23}E_3) \wedge \theta_2 + E_2 d\theta_2 = 0 \tag{5}$$

From (5) there immediately follows

$$(d\theta_1 + \omega_{21} \wedge \theta_2)E_1 + (d\theta_2 + \omega_{12} \wedge \theta_1)E_2 + (\omega_{13} \wedge \theta_1 + \omega_{23} \wedge \theta_2)E_3 = 0. \tag{6}$$

The linear independence of vectors E_1, E_2, E_3 and equation (6) gives the following equations:

$$d\theta_1 = \omega_{12} \wedge \theta_2, \tag{7}$$

$$d\theta_2 = \omega_{21} \wedge \theta_1, \tag{8}$$

$$0 = \omega_{13} \wedge \theta_1 + \omega_{23} \wedge \theta_2. \tag{9}$$

Differentiating (3) gives:

$$\begin{aligned} &0 = d^2E_1 = d\omega_{12}E_2 - \omega_{12} \wedge dE_2 + d\omega_{13}E_3 - \omega_{13} \wedge dE_3, \\ &d\omega_{12}E_2 - \omega_{12} \wedge (\omega_{21}E_1 + \omega_{23}E_3) + d\omega_{13}E_3 - \omega_{13} \wedge (\omega_{31}E_1 + \omega_{32}E_2) = 0, \\ &(d\omega_{12} - \omega_{13} \wedge \omega_{32})E_2 + (d\omega_{13} - \omega_{12} \wedge \omega_{23})E_3 = 0. \end{aligned} \tag{10}$$

From (10) we have

$$d\omega_{12} = \omega_{13} \wedge \omega_{32} \tag{11}$$

$$d\omega_{13} = \omega_{12} \wedge \omega_{23} \tag{12}$$

Analogically:

$$\begin{aligned} d^2E_2 &= d\omega_{21}E_1 - \omega_{21} \wedge dE_1 + d\omega_{23}E_3 - \omega_{23} \wedge dE_3 \quad = 0. \\ d\omega_{21}E_1 - \omega_{21} \wedge \left(\omega_{12}E_2 + \omega_{13}E_3\right) &+ d\omega_{23}E_3 - \omega_{23} \wedge \left(\omega_{31}E_1 + \omega_{32}E_2\right) = 0. \\ \left(d\omega_{23} - \omega_{21} \wedge \omega_{13}\right)E_3 &+ \left(d\omega_{21} - \omega_{23} \wedge \omega_{31}\right)E_1 \quad = 0 \end{aligned} \tag{13}$$

From (13) we have

$$d\omega_{23} = \omega_{21} \wedge \omega_{13}. \tag{14}$$

Equations (7), (8), (9), (11), (12) and (14) are called Maurer-Cartan structural equations. From equation (9) and Cartan's lemma we have

$$\omega_{13} = \alpha_{11}\theta_1 + \alpha_{12}\theta_2 \quad , \quad \omega_{23} = \alpha_{12}\theta_1 + \alpha_{22}\theta_2. \tag{15}$$

From (15) and (11) we have

$$d\omega_{12} = \omega_{13} \wedge \omega_{32} = -\omega_{13} \wedge \omega_{23} = -\left(\alpha_{11}\theta_1 + \alpha_{12}\theta_2\right) \wedge \left(\alpha_{12}\theta_1 + \alpha_{22}\theta_2\right). \tag{16}$$

Equation (16) gives

$$d\omega_{12} = -\left(\alpha_{11}\alpha_{22} - \alpha_{12}^2\right)\theta_1 \wedge \theta_2 = -K\theta_1 \wedge \theta_2 \,,$$

where $K = \left(\alpha_{11}\alpha_{22} - \alpha_{12}^2\right)$ is the Gaussian curvature.

Differentiating the equation $E_3 \cdot E_3 = 1$ we obtain $dE_3 \cdot E_3 = 0$, which means that dE_3 is a tangent vector, i.e. $dE_3 \in T_p(M)$. The mapping

$$W\left(\alpha x_u + \beta x_v\right) = -\alpha \frac{\partial E_3}{\partial u} - \beta \frac{\partial E_3}{\partial v} \tag{17}$$

is a linear mapping $W : T_p(M) \to T_p(M)$.

2. Example 1

Let $x(u,v) = (u, v, u \cdot v)$ be the parametrized utility surface in R^3. Orthogonal frame is

$$x_u = (1,0,v), \quad x_v = (0,1,u), \quad n = (-v,-u,1).$$

Orthonormal frame is

$$E_1 = \frac{1}{\sqrt{1+v^2}}(1,0,v),$$
$$E_2 = \frac{1}{\sqrt{1+v^2}\cdot\sqrt{1+u^2+v^2}}(-uv,1+v^2,u),$$
$$E_3 = \frac{1}{\sqrt{1+u^2+v^2}}(-v,-u,1).$$

The differential form $dx = x_u du + x_v dv$ gives

$$\theta_i = E_i dx = E_i x_u du + E_i x_v dv, \quad \text{for} \quad i = 1,2.$$

We have

$$\theta_1 = \sqrt{1+v^2}\,du + \frac{uv}{\sqrt{1+v^2}}\,dv, \tag{18}$$
$$\theta_2 = \frac{\sqrt{1+u^2+v^2}}{\sqrt{1+v^2}}\,dv. \tag{19}$$

Further we have

$$dE_1 = \left(\frac{-v}{\left(1+v^2\right)^{\frac{3}{2}}} \ , \ 0 \ , \ \frac{1}{\left(1+v^2\right)^{\frac{3}{2}}} \right) dv,$$
$$\omega_{12} = dE_1 \cdot E_2 = \frac{u}{\left(1+v^2\right)\sqrt{1+u^2+v^2}}\,dv.$$

Analogically we have

$$\omega_{13} = dE_1 \cdot E_3 = \frac{1}{\sqrt{1+v^2}\sqrt{1+u^2+v^2}}\,dv.$$

And further

$$\partial_u E_2 = \frac{\sqrt{1+v^2}}{\left(1+u^2+v^2\right)^{\frac{3}{2}}} \cdot (-v, -u,\ 1), \tag{20}$$

$$\partial_v E_2 = \frac{1}{\left(1+v^2\right)^{\frac{3}{2}} \cdot \left(1+u^2+v^2\right)^{\frac{3}{2}}} \left(E_{2v}^1, E_{2v}^2, E_{2v}^3\right), \tag{21}$$

where

$$\begin{aligned} E_{2v}^1 &= -u\left(1+v^2\right)\left(1+u^2\right) + uv^2\left(1+u^2+v^2\right), \\ E_{2v}^2 &= u^2 v\left(1+v^2\right), \\ E_{2v}^3 &= -uv\left[\left(1+u^2+v^2\right) + \left(1+v^2\right)\right]. \end{aligned}$$

From (20) and (21) follows

$$\partial_u E_2 \cdot E_3 = \frac{\sqrt{1+v^2}}{1+u^2+v^2} \quad , \quad \partial_v E_2 \cdot E_3 = \frac{-uv}{\sqrt{1+v^2}\left(1+u^2+v^2\right)},$$

and

$$\omega_{23} = dE_2 \cdot E_3 = \frac{\sqrt{1+v^2}}{1+u^2+v^2} du - \frac{uv}{\sqrt{1+v^2}\left(1+u^2+v^2\right)} dv.$$

Summarizing the previous results, we have

$$\begin{aligned} \omega_{12} &= -\omega_{21} = \frac{u\, dv}{\left(1+v^2\right)\sqrt{1+u^2+v^2}}, \\ \omega_{13} &= -\omega_{31} = \frac{dv}{\sqrt{1+v^2}\sqrt{1+u^2+v^2}}, \\ \omega_{23} &= -\omega_{32} = \frac{\sqrt{1+v^2}}{1+u^2+v^2} du - \frac{uv}{\sqrt{1+v^2}\left(1+u^2+v^2\right)} dv, \\ \theta_1 &= \sqrt{1+v^2}\, du + \frac{uv}{\sqrt{1+v^2}} dv, \\ \theta_2 &= \frac{\sqrt{1+u^2+v^2}}{\sqrt{1+v^2}} dv. \end{aligned}$$

From equations (20) and (21) follows

$$d\theta_1 = 0 \quad , \quad d\theta_2 = \frac{u}{\sqrt{1+v^2}\sqrt{1+u^2+v^2}} du \wedge dv.$$

And

$$\theta_1 \wedge \theta_2 = \sqrt{1+u^2+v^2} du \wedge dv. \tag{22}$$

From (11) we have

$$d\omega_{12} = \omega_{13} \wedge \omega_{32} = \frac{1}{\left(1+u^2+v^2\right)^{\frac{3}{2}}} du \wedge dv.$$

Thanks to (22) we have

$$du \wedge dv = \frac{1}{\sqrt{1+u^2+v^2}} \theta_1 \wedge \theta_2$$

and

$$d\omega_{12} = \frac{1}{\left(1+u^2+v^2\right)^2} \theta_1 \wedge \theta_2. \tag{23}$$

From (23) immediately follows that

$$K = -\frac{1}{\left(1+u^2+v^2\right)^2}, \tag{24}$$

which means that every point of studied surface is hyperbolical. The equation (17) gives

$$W(x_u) = -\partial_u E_3 \quad \text{and} \quad W(x_v) = -\partial_v E_3,$$

where

$$\partial_u E_3 = \frac{1}{\left(1+u^2+v^2\right)^{\frac{3}{2}}}\left(uv, -v^2-1, -u\right) \quad , \quad \partial_v E_3 = \frac{1}{\left(1+u^2+v^2\right)^{\frac{3}{2}}}\left(1-u^2, uv, -v\right).$$

From the fact $W : T_p(M) \rightarrow T_p(M)$ follows

$$\partial_u E_3 = \alpha_{11} x_u + \alpha_{12} x_v \quad , \quad \partial_v E_3 = \alpha_{21} x_u + \alpha_{22} x_v . \tag{25}$$

After a short calculation we obtain

$$\alpha_{11} = \frac{uv}{\left(1+u^2+v^2\right)^{\frac{3}{2}}}, \quad \alpha_{12} = -\frac{1+v^2}{\left(1+u^2+v^2\right)^{\frac{3}{2}}},$$
$$\alpha_{21} = -\frac{1+u^2}{\left(1+u^2+v^2\right)^{\frac{3}{2}}}, \quad \alpha_{22} = \frac{uv}{\left(1+u^2+v^2\right)^{\frac{3}{2}}}.$$

From equations (25) follows that the mapping W can be described by the matrix

$$W = \frac{1}{\left(1+u^2+v^2\right)^{\frac{3}{2}}} \begin{pmatrix} -uv & 1+v^2 \\ 1+u^2 & -uv \end{pmatrix}.$$

Determinant

$$\det\ W = K = \frac{1}{\left(1+u^2+v^2\right)^{3}} \det \begin{pmatrix} -uv & 1+v^2 \\ 1+u^2 & -uv \end{pmatrix} = -\frac{1}{\left(1+u^2+v^2\right)^{2}} \ ,$$

as was given in (24) and the formula for mean curvature

$$H = \frac{1}{2} Tr\, W = -\frac{uv}{\left(1+u^2+v^2\right)^{\frac{3}{2}}}.$$

3. Example 2

Let $x(u,v) = (u, v, u^2 v)$ be a parameterized utility function. Orthogonal frame is

$$x_u = (1,\ 0,\ 2uv)$$
$$x_v = (0,\ 1,\ u^2)$$
$$n = (-2uv,\ -u^2,\ 1).$$

Orthonormal frame is

$$E_1 = \frac{1}{\sqrt{1+4u^2v^2}} (1,\ 0,\ 2uv),$$
$$E_2 = \frac{1}{\sqrt{1+4u^2v^2} \cdot \sqrt{1+4u^2v^2+u^4}} (-2u^3 v, 1+4u^2v^2,\ u^2), \tag{26}$$
$$E_3 = \frac{1}{\sqrt{1+4u^2v^2+u^4}} (-2uv,\ -u^2,\ 1).$$

The forms θ_1 and θ_2 have the form

$$\theta_1 = \sqrt{1+4u^2v^2}\,du + \frac{2u^3v}{\sqrt{1+4u^2v^2}}\,dv,\ \theta_2 = \frac{\sqrt{1+4u^2v^2+u^4}}{\sqrt{1+4u^2v^2}}\,dv. \tag{27}$$

Further we have

$$dE_1 = \partial_u E_1 du + \partial_v E_1 dv = \frac{1}{(1+4u^2v^2)^{3/2}}[(-4uv^2,\ 0,\ 2v)du + (-4u^2v,\ 0,\ 2u)\ dv].$$

After a short calculation we obtain

$$\omega_{12} = dE_1 \cdot E_2 = \frac{1}{(1+4u^2v^2)\sqrt{1+4u^2v^2+u^4}}(2u^2v\,du + 2u^3\ dv).$$

Analogically

$$\omega_{13} = dE_1 \cdot E_3 = \frac{1}{\sqrt{1+4u^2v^2}\sqrt{1+4u^2v^2+u^4}}(2v\,du + 2u\,dv).$$

From (26) follows

$$dE_3 = (\partial_u E_3)du + (\partial_v E_3)dv.$$

After a short calculation we obtain

$$d\omega_{32} = dE_3 \cdot E_2 = \frac{1}{\sqrt{1+4u^2v^2}(1+4u^2v^2+u^4)}[(-4u^3v^2 - 2u)\,du + 4u^4v\,dv].$$

Summarizing the previous results we obtain

$$\omega_{12} = -\omega_{21} = \frac{1}{(1+4u^2v^2)\sqrt{1+4u^2v^2+u^4}}\left(2u^2\mathrm{v}\,du + 2u^3\ dv\right),$$
$$\omega_{13} = -\omega_{31} = \frac{1}{\sqrt{1+4u^2v^2}\sqrt{1+4u^2v^2+u^4}}\left(2v\,du + 2u\,dv\right),$$
$$\omega_{23} = -\omega_{32} = \frac{1}{\sqrt{1+4u^2v^2}\left(1+4u^2v^2+u^4\right)}\left[\left(4u^3v^2+2u\right)du - 4u^4v\,dv\right].$$

From equations (16) and (27) we obtain

$$d\omega_{12} = \omega_{13} \wedge \omega_{32} = \frac{4u^2}{\left(1+4u^2v^2+u^4\right)^{\frac{3}{2}}}du \wedge dv = \frac{4u^2}{\left(1+4u^2v^2+u^4\right)^2}\theta_1 \wedge \theta_2.$$

from which follows: Gaussian curvature has the form $K = -\frac{4u^2}{\left(1 + 4u^2v^2 + u^4\right)^2}$.

Conclusion

Two economical examples served as an illustration of Maurer-Cartan equations and we reached the following results:

1. The Gaussian and mean curvatures of the first surface are

$$K = -\frac{1}{\left(1 + u^2 + v^2\right)^2}, \quad H = \frac{1}{2} Tr\, W = -\frac{uv}{\left(1 + u^2 + v^2\right)^{\frac{3}{2}}}.$$

2. The Gaussian curvature of the second surface is $K = -\frac{4u^2}{\left(1 + 4u^2v^2 + u^4\right)^2}$.

References

Bureš, Jarolím, and Kaňka, Miloš. "Some Conditions for a Surface in E^4 to be a Part of the Sphere S^2." *Mathematica Bohemica* 4 1994: 367-371.

Cartan, Élie. *Euvres complètes-Partie I Vol. 2.* Paris: Gauthier-Villars, 1952.

Cartan, Élie. *Euvres complètes-Partie II. Vol. 1.* Paris: Gauthier-Villars, 1953.

Cartan, Élie. *Euvres complètes-Partie II. Vol. 2.* Paris: Gauthier-Villars, 1953.

Cartan, Élie. *Euvres complètes-Partie III. Vol. 1.* Paris: Gauthier-Villars, 1955.

Cartan, Élie. *Euvres complètes-Partie III. Vol. 2.* Paris: Gauthier-Villars, 1955.

Kaňka, Miloš. "Example of Basic Structure Equations of Riemannian Manifolds." *Mundus Symbolicus* 3 1995: 57-62.

Kobayashi, Shoshichi, and Nomizu, Katsumi. *Foundations of Differential Geometry.* New York: Wiley (Inter-science), 1963.

Nomizu, Katsumi. *Lie Groups and Differential Geometry.* Tokyo: The Mathematical Society of Japan, 1956.

Sternberg, Sholomo. *Lectures on Differential Geometry.* Englewood Cliffs, N.J.: Prentice-Hall, 1964.

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