MATHEMATICAL STUDIES OF SOME EXAMPLES OF PRODUCTION FUNCTIONS

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Abstract

The aim of this paper is to give some examples of generalized Cobb-Douglas surfaces and some geometric characteristics of these surfaces. In case of growing returns to scale Cobb-Douglas surfaces have the form

$$\gamma(x,y)=(x,y,A\cdot x^{\alpha}\cdot y^{\beta}),\ \text{where}\ \ x>0,\ \ y>0,\ \ \alpha+\beta>1,\ \ \alpha>0,\ \ \beta>0.$$

In case of decrease returns to scale Cobb-Douglas surfaces have the form

$$\gamma(x,y)=(x,y,A\cdot x^{\alpha}\cdot y^{\beta}),\ \text{where}\ \ x>0,\ \ y>0,\ \alpha+\beta<1,\ \ \alpha>0,\ \ \beta>0.$$

Analogically in case of constant returns to scale Cobb-Douglas surfaces have the form

$$\gamma(x,y)=(x,y,A\cdot x^{\alpha}\cdot y^{\beta}),\text{where}\ \ x>0,\ \ y>0,\ \alpha+\beta=1,\ \ \alpha>0,\ \ \beta>0.$$

We are interested in Gaussian curvature, mean curvature and principal curvatures of these surfaces.

Key words: Tangent vectors, normal vector, Weingarten map, Gaussian curvature, Mean curvature, first and second fundamental form, shape operator.

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Surfaces in R^3

A subset $S\subset R^3$ is a regular surface if for every point $p\in S$ there is an open set $U\subset R^2$ and an open set $V\subset R^3$, $p\in V$ such that there is a regular map $\gamma:U\to\ R^3$ which is a homeomorphism of an open set $U\subset R^2$ onto open set $V\cap S$.

A surface map $\gamma:U\to S\subset R^3$ where U is an open set in R^2 is called regular if it is smooth and tangent vectors γ_x and γ_y are linearly independent at all points $(x,y)\in U$, such that the normal vector $\gamma_x\times\gamma_y=N$ is a non-vanishing vector field on a regular surface S, everywhere perpendicular to S. A regular surface is a surface $S\subset R^3$ atlas of which consists of regular maps. In this paper the notion map always means a regular map and we are to study smooth surfaces, whose atlas consists of regular maps. The basic tool for our study is the shape operator defined as follows.

Definition 1. Let $S \subset R^3$ be a regular surface and let n be a surface unit normal to S defined in a neighbourhood of a point $x \in S.$ For a tangent vector $v \in T_x(S)$ we define $\varphi(v) = -n_v.$

Lemma 1. Let $S \subset R^2$ and $\gamma : U \to R^3$ be a regular map. Then

$$\varphi(\gamma_x) = -n_x \quad \text{and} \quad \varphi(\gamma_y) = -n_y. \tag{1}$$

Proof: For fix y_0, $\gamma(x, y_0)$ is a curve in S. We have

$$\varphi(\gamma_x(x, y_0)) = \varphi(\gamma'(x, y_0)) = -n_{\gamma'(x,y_0)} = -(n \circ \gamma)'(x) = -n_x.$$

Analogically $\varphi(\gamma_y(x_0, y)) = -n_y.$

Lemma 2. At each point x of a regular surface $S \subset R^3$, the shape operator is a linear map $\varphi : T_x(S) \to T_x(S).$

Remark 1. Let γ be a regular injective map. The equations $n \cdot \gamma_x = 0$ and $n \cdot \gamma_y = 0$ give

$$\begin{aligned}
0 &= (n \cdot \gamma_x)_x = n_x \cdot \gamma_x + n \cdot \gamma_{xx} \Rightarrow -n_x\gamma_x = n \cdot \gamma_{xx}, \\
0 &= (n \cdot \gamma_x)_y = n_y \cdot \gamma_x + n \cdot \gamma_{xy} \Rightarrow -n_y\gamma_x = n \cdot \gamma_{xy} = n \cdot \gamma_{yx}, \\
0 &= (n \cdot \gamma_y)_y = n_y \cdot \gamma_y + n \cdot \gamma_{yy} \Rightarrow -n_y\gamma_y = n \cdot \gamma_y.
\end{aligned}$$

Remark 2. Let $\gamma : U \to R^3$ be a regular injective map. Let us denote

$$\begin{aligned}
l_{11} &= -n_x \cdot \gamma_x = n\gamma_{xx}, \\
l_{12} &= -n_y \cdot \gamma_x = n\gamma_{xy} = n\gamma_{yx} = -n_x\gamma_y, \\
l_{22} &= -n_y \cdot \gamma_y = n\gamma_{yy}.
\end{aligned} \tag{2}$$

The functions l_{11}, l_{12}, l_{22} are coefficients of the second fundamental form F_{II} of γ.

$$F_{II} = l_{11}dx^2 + 2l_{12}dxdy + l_{22}dy^2.$$

If we denote $g_{11} = \|\gamma_x\|^2$, $g = \gamma_x \cdot \gamma_y$, $g_{22} = \|\gamma_y\|^2$, the first fundamental form F_I can be written in the form

$$F_I = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2.$$

Theorem 1. Let $\gamma : U \to R^3$ be a regular injective map. Then the shape operator φ is given with respect to the basis $\gamma_x, \gamma_y \in T_x(S)$ in the form

$$\begin{aligned}\varphi(\gamma_x) &= \frac{g_{22}l_{11} - g_{12}l_{12}}{g_{11}g_{22} - g_{12}^2}\gamma_x + \frac{g_{11}l_{12} - g_{12}l_{11}}{g_{11}g_{22} - g_{12}^2}\gamma_y,\\ \varphi(\gamma_y) &= \frac{g_{22}l_{12} - g_{12}l_{22}}{g_{11}g_{22} - g_{12}^2}\gamma_x + \frac{g_{11}l_{22} - g_{12}l_{12}}{g_{11}g_{22} - g_{12}^2}\gamma_y.\end{aligned} \tag{3}$$

Proof: As γ is a regular injective map and γ_x and γ_y are linearly independent we have

$$\begin{aligned}\varphi(\gamma_x) &= \alpha_{11}\gamma_x + \alpha_{21}\gamma_y = -n_x,\\ \varphi(\gamma_y) &= \alpha_{12}\gamma_x + \alpha_{22}\gamma_y = -n_y,\end{aligned} \tag{4}$$

for functions $\alpha_{11}, \alpha_{21}, \alpha_{12}, \alpha_{22}$ which we need to compute. From (1) and (3) we have

$$\begin{aligned}l_{11} &= -n_x\gamma_x = g_{11}\alpha_{11} + g_{12}\alpha_{21},\\ l_{12} &= -n_x\gamma_y = g_{12}\alpha_{11} + g_{22}\alpha_{21},\\ l_{12} &= -n_y\gamma_x = g_{11}\alpha_{12} + g_{12}\alpha_{22},\\ l_{22} &= -n_y\gamma_y = g_{12}\alpha_{12} + g_{22}\alpha_{22}.\end{aligned} \tag{5}$$

Equations (5) can be written in the form

$$\begin{pmatrix} l_{11} & l_{12} \\ l_{12} & l_{22} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$$

or

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}^{-1}\begin{pmatrix} l_{11} & l_{12} \\ l_{12} & l_{22} \end{pmatrix}.$$

So we have

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \frac{1}{g_{11}g_{22} - g_{12}^2}\begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix}\begin{pmatrix} l_{11} & l_{12} \\ l_{12} & l_{22} \end{pmatrix}.$$

From which immediately follows (2).

Remark 3. The shape operator can be represented by a matrix

$$A(\varphi) = \begin{pmatrix} \dfrac{g_{22}l_{11} - g_{12}l_{12}}{g_{11}g_{22} - g_{12}^2} & \dfrac{g_{11}l_{12} - g_{12}l_{11}}{g_{11}g_{22} - g_{12}^2} \\ \dfrac{g_{22}l_{12} - g_{12}l_{22}}{g_{11}g_{22} - g_{12}^2} & \dfrac{g_{11}l_{22} - g_{12}l_{12}}{g_{11}g_{22} - g_{12}^2} \end{pmatrix}.$$

As $K = \det A(\varphi)$ and $H = \frac{1}{2}\operatorname{tr}A(\varphi)$ we have

$$K = \frac{l_{11}l_{22} - l_{12}^2}{g_{11}g_{22} - g_{12}^2} \qquad \text{and} \qquad H = \frac{l_{11}g_{22} - 2l_{12}g_{12} + l_{22}g_{11}}{2(g_{11}g_{22} - g_{12}^2)}.$$

Examples

Example 1

In case of growing returns to scale we will study the Gaussian curvature, mean curvature and principal curvatures of a special type Cobb-Douglas surface of the form

$$\gamma(x, y) = (x, y, xy), \text{ i.e. } \alpha + \beta = 2 \text{ (see Fig. 1).}$$

Solution. We have

$$\gamma_x = (1,0, y),\ \gamma_y = (0,1, x),\ g_{11} = 1 + y^2,\ g_{12} = xy,\ g_{22} = 1 + x^2.$$

The unit normal is $n = \frac{(-y,-x,1)}{\sqrt{x^2 + y^2 + 1}}$. Further we have

$$\gamma_{xx} = (0,0,0), \quad \gamma_{xy} = (0,0,1), \quad \gamma_{yy} = (0,0,0).$$

The equations

$$l_{11} = n \cdot \gamma_{xx}, \quad l_{12} = n \cdot \gamma_{xy}, \quad l_{22} = n \cdot \gamma_{yy}.$$

give

$$l_{11} = 0,\ l_{12} = \frac{1}{\sqrt{x^2 + y^2 + 1}},\ l_{22} = 0 .$$

So we have

$$K = \frac{-1}{(x^2 + y^2 + 1)^2},\ H = \frac{-xy}{(x^2 + y^2 + 1)^{3/2}}.$$

From the first of the previous equations follows that for all $x, y \in R$ the Gaussian curvature is negative, which means that every point of Cobb-Douglas surface $\gamma(x, y) = (x, y, xy)$ is hyperbolical. Principal curvatures k_1 and k_2 can be written in the form:

$$k_1 = \frac{1}{(x^2+y^2+1)^{3/2}}\left[-xy+\sqrt{x^2y^2+(x^2+y^2+1)}\right]$$

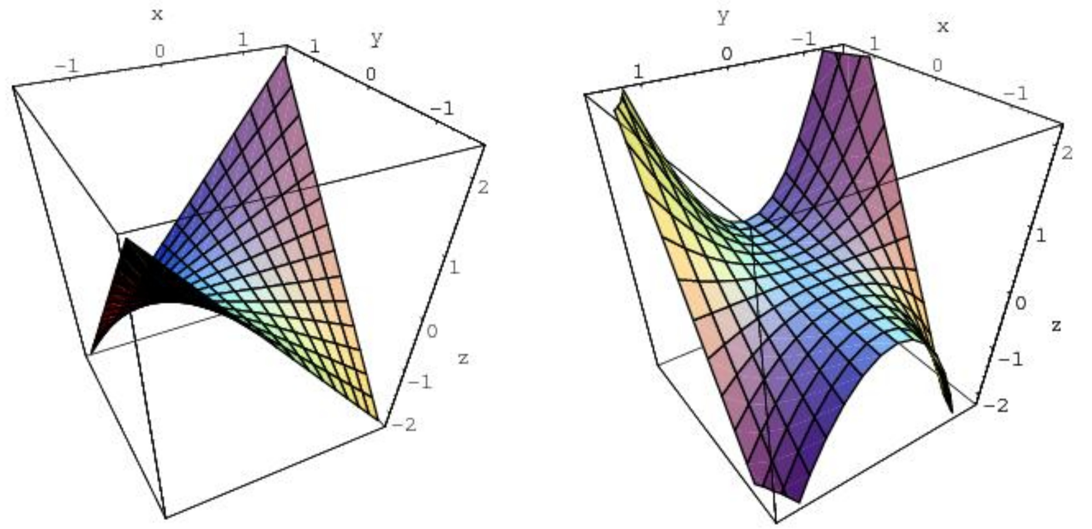
and $$k_2 = \frac{1}{(x^2+y^2+1)^{3/2}}\left[-xy-\sqrt{x^2y^2+(x^2+y^2+1)}\right].$$

Example 2

In this example we will study Gaussian curvature, mean curvature and principal curvatures of Cobb-Douglas surface

$$\gamma(x,y)=(x,y,xy^2),\ \text{i.e.}\ \alpha+\beta=3\ \text{(see Fig. 2).}$$

Fig. 1



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Solution. The basis of $T_x(S)$ has the form $\gamma_x=(1,0,y^2)$, $\gamma_y=(0,1,2xy)$,

$$g_{11}=1+y^4,\qquad g_{12}=2xy^3,\qquad g_{22}=1+4x^2y^2,$$
$$\gamma_{xx}=(0,0,0),\quad \gamma_{xy}=(0,0,2y),\quad \gamma_{yy}=(0,0,2x).$$

The unit normal is $n=\dfrac{(-y^2,-2xy,1)}{\sqrt{y^4+4x^2y^2+1}}$. Further, we have

$$l_{11}=0,\quad l_{12}=\frac{2y}{\sqrt{y^4+4x^2y^2+1}},\quad l_{22}=\frac{2x}{\sqrt{y^4+4x^2y^2+1}}.$$

The Gaussian curvature and mean curvature have the forms

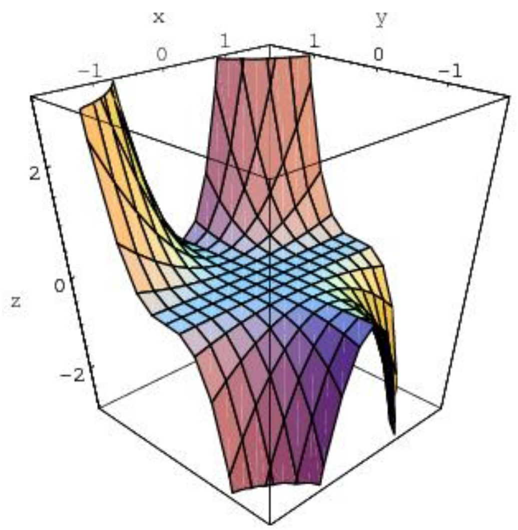
$$K = \frac{-4y^2}{(y^4 + 4x^2y^2 + 1)^2} \le 0, \qquad H = \frac{x - 3xy^4}{(y^4 + 4x^2y^2 + 1)^{3/2}}.$$

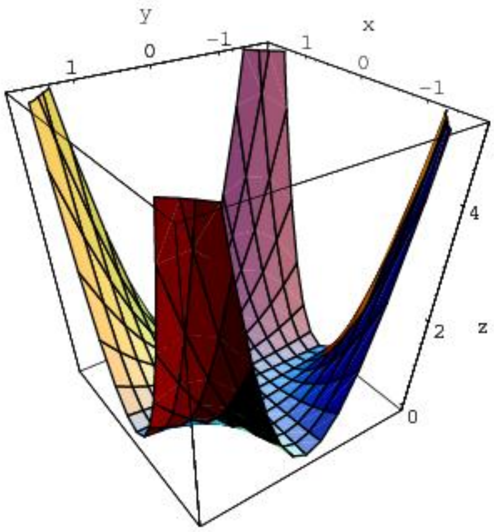
If $y \neq 0$ then $K < 0$ and every point of this type of given surface is hyperbolical. Principal curvatures can be written in the form:

$$k_1 = \frac{1}{(y^4 + 4x^2y^2 + 1)^{3/2}} \left[x - 3xy^4 + \sqrt{(x - 3xy^4)^2 + 4y^2(y^4 + 4x^2y^2 + 1)} \right],$$

$$k_2 = \frac{1}{(y^4 + 4x^2y^2 + 1)^{3/2}} \left[x - 3xy^4 - \sqrt{(x - 3xy^4)^2 + 4y^2(y^4 + 4x^2y^2 + 1)} \right].$$

Fig. 2





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Example 3

In this example we will study Cobb-Douglas surfaces of the form:

$$1) \quad \gamma(x, y) = (x, y, x^2y^2), \text{ i.e. } \alpha + b = 4 \text{ (see Fig. 3)}$$

Solution. The basis of $T_x(S)$ has the form $\gamma_x = (1,0,2xy^2)$, $\gamma_y = (0,1,2yx^2)$,

$$g_{11} = 1 + 4x^2y^4, \; g_{12} = 4x^3y^3, \; g_{22} = 1 + 4x^4y^2.$$

The unit normal has the form $n = \frac{(-2xy^2, -2yx^2, 1)}{\sqrt{4x^2y^4 + 4y^2x^4 + 1}}$. Functions l_{11}, l_{12}, l_{22} are:

$$l_{11} = \frac{2y^2}{\lambda}, \; l_{12} = \frac{4xy}{\lambda}, \; l_{22} = \frac{2x^2}{\lambda}, \text{ where } \lambda = \sqrt{4x^2y^4 + 4y^2x^4 + 1}.$$

The Gaussian curvature and mean curvature can be written in the form

$$K = \frac{-12x^2y^2}{(4x^2y^4 + 4y^2x^4 + 1)^2}, \quad H = \frac{x^2 + y^2 - 8x^4y^4}{(4x^2y^4 + 4y^2x^4 + 1)^{3/2}}.$$

$K \le 0$ for all $(x, y) \in R.$ Every point $(x, y) \ne (0,0)$ is hyperbolical.

2) $\gamma(x, y) = (x, y, x^2y^3)$, i.e. $\alpha + b = 5$ (see Fig. 4)

Solution. The basis of tangent space has the form $\gamma_x = (1, 0, 2xy^3)$, $\gamma_y = (0, 1, 3x^2y^2)$. Functions g_{11}, g_{12}, g_{22} are

$$g_{11} = 1 + 4x^2y^6, \; g_{12} = 6x^3y^5, \; g_{22} = 1 + 9x^4y^4.$$

We have $\gamma_{xx} = (0,0,2y^3)$, $\gamma_{xy} = (0,0,6xy^2)$, $\gamma_{yy} = (0,0,6x^2y)$. The unit normal has the form $n = \frac{(-2xy^3, -3x^2y^2, 1)}{(4x^2y^6 + 9x^4y^4 + 1)^{1/2}}$. The functions l_{11}, l_{12}, l_{22} are

$$l_{11} = \frac{2y^3}{\lambda}, \; l_{12} = \frac{6xy^2}{\lambda}, \; l_{22} = \frac{6x^2y}{\lambda}, \text{ where } \lambda = (4x^2y^6 + 9y^4x^4 + 1)^{1/2}.$$

The Gaussian curvature and mean curvature are

$$K = \frac{-24x^2y^4}{(4x^2y^6 + 9x^4y^4 + 1)^2}, \quad H = \frac{y^3 + 3x^2y - 15x^4y^7}{(4x^2y^6 + 9x^4y^4 + 1)^{3/2}}.$$

So we have $K \le 0$ and if $(x, y) \ne (0,0)$ then $K < 0$. Every point of given surface for which $(x, y) \ne (0,0)$ is hyperbolical.

Example 4

In this example we will study the general case of Cobb-Douglas surface

$$\gamma(x, y) = (x, y, x^m \cdot y^n), \text{ where } m, n \text{ are constants, } m > 0, n > 0.$$

Solution. The basis of tangent space can be written in the form

$$\gamma_x = (1,0, m \cdot x^{m-1} y^n), \quad \gamma_y = (0,1, n \cdot y^{n-1} x^m).$$

Functions g_{11}, g_{12}, g_{22} have the form

$$g_{11} = 1 + m^2 x^{2m-2} y^{2n}, \quad g_{12} = m \cdot n \cdot x^{2m-1} \cdot y^{2n-1}, \quad g_{22} = 1 + n^2 y^{2n-2} \cdot x^{2m}.$$

The unit normal has the form

$$n = \frac{(-mx^{m-1} y^n, \quad -ny^{n-1} \cdot x^m, \quad 1)}{\lambda^{1/2}},$$

where $\lambda = m^2 x^{2m-2} y^{2n} + n^2 y^{2n-2} \cdot x^{2m} + 1.$

$$\gamma_{xx} = (0,0, m(m-1) x^{m-2} \cdot y^n),$$
$$\gamma_{xy} = (0,0, m \cdot n x^{m-1} \cdot y^{n-1}),$$
$$\gamma_{yy} = (0,0, n(n-1) y^{n-2} \cdot x^m).$$

The functions l_{11}, l_{12}, l_{22} can be written in the form

$$l_{11} = \frac{m(m-1) x^{m-2} y^n}{\sqrt{\lambda}},$$
$$l_{12} = \frac{m \cdot n \cdot x^{m-1} y^{n-1}}{\sqrt{\lambda}},$$
$$l_{22} = \frac{n(n-1) y^{n-2} x^m}{\sqrt{\lambda}}.$$

The Gaussian curvature has the form

$$K = \frac{mn \cdot [1 - (m+n)] \cdot x^{2m-2} y^{2n-2}}{(m^2 x^{2m-2} y^{2n} + n^2 x^{2m} y^{2n-2} + 1)^2}. \tag{6}$$

Conclusion

From formula (6) can be easily seen that following implications are true:

1) $m + n > 1 \Rightarrow K \le 0,$ and if $(x, y) \ne (0,0) \Rightarrow K < 0,$
   which means that every point (x, y) where $(x, y) \ne (0,0)$ of Cobb-Douglas surface is hyperbolical and principal curvatures k_1 and k_2 have the opposite signs.

2) $m + n = 1 \Rightarrow K = 0,$
   which means that every point of Cobb-Douglas surface is parabolic. Exactly one of principal curvatures is zero.

3) $m+n<1 \Rightarrow K \geq 0$, and if $(x,y) \neq (0,0) \Rightarrow K>0$,
which means that every point (x,y) where $(x,y) \neq (0,0)$ is elliptical and principal curvatures k_1 and k_2 have the same sign.

The mean curvature can be written in the form

$$H=\frac{m(m-1)x^{m-2}y^{n}+n(n-1)y^{n-2}x^{m}-mn(m+n)x^{3m-2}y^{3n-2}}{2(m^2x^{2m-2}y^{2n}+n^2x^{2m}y^{2n-2}+1)^{3/2}}.$$

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