PCA FACTOR MODEL FOR FORWARD EURO EXPOSURE

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Abstract

The measurement of risk of the portfolios has a very important role in risk management. We use the PCA factor model to decompose the risk from a sequence of forward foreign currency payments into interest rate, exchange rate and correlation components. In this paper we focus on the application of PCA to forward currency exposures and futures positions in commodities. The contract will be related to British importer purchasing grain or another commodity in euro. The euro price of grain has been fixed and there is no commodity price risk. In this paper we show how to decompose the total risk into three components, namely into the exchange rate risk that arises from uncertainty about the British pound value of future payments in euro, the interest rate risk that arises from the change in present value of the British pound cash flows and the correlation risk that arises from British exchange rates and the euro-dollar exchange rates and how to isolate the key interest rate risk factors.

Key words: Principal Component Analysis, portfolio risk management, Factor model

JEL Code: G12, G32, C80, C58

Introduction

New risk capital regulations well–established by Basel Accord Amendment in 1996 emphasize the requirement to measure and quantify risks of the financial portfolios. Current problems produced in the financial sector, indicate the need to quantify the risks associated with investing in financial portfolios. For risk managers it is an enormous advantage to be able to know key risk factors and to quantify how much risk is associated with each key factor. Their attention is then more easily directed towards the most important sources of risk. The financial portfolios consist of multiple assets, and their returns depend concurrently and dynamically on many economic and financial variables. In practice, the return of the economic and financial variables often exhibits similar characteristics leading to the belief that it might be driven by some common sources, often referred to as common factors. These common factors can be determined by using principal component analysis (PCA). Jamshidian and Zhu (1997) have shown how the PCA may be used to improve computational efficiency for scenario based risk measures in large multi–currency portfolios.

In this paper we focus on the application of PCA to forward currency exposures and futures positions in commodities. The contract will be related to UK importer purchasing grain or another commodity in euro. So the euro price of grain has been fixed and there is no commodity price risk. However, the risks remaining are:

1. the exchange rate risk, arising from uncertainty about the sterling value of future payments in euro;
2. the interest rate risk, arising from the change in present value of the sterling cash flows;
3. the correlation risk, arising from the correlation between UK interest rates and the euro–dollar exchange rates.

In this paper we show how to decompose the total risk into these three components and how to isolate the key interest rate risk factors.

The rest of the paper is organized as follows. Section 1 describes the methodology employed. Section 2 describes the dataset, represents and discusses the empirical results obtained. Section 3 finally concludes and suggests relevant implications based on the empirical findings.

1 Principal component factor model representation and decomposing of the risk for the forward positions in commodities

The future positions in commodities can be characterized by a cash flow, i.e. a series of cash payments $\boldsymbol{C} = \{ C_{T_1}, \ldots, C_{T_n} \}$ at different times T_i, for $i = 1, 2, \ldots n$ in future. For this reason the market risk of this portfolio is usually analysed by mapping all the cash flows to a fixed, finite set of zero coupon yields at standard maturities such as 1 month, 2 month, and so on up to the maximum maturity of the instruments. Such a set of discount rates is a set of risk factors for the portfolio.

Mapping of this portfolio to a finite set of zero coupon yields requires calculation of the net sensitivities of the portfolio to each of these risk factors. The net sensitivities are measured by the present value of a basis point move ($PV01$[1]). The changes in this portfolio

[1] $PV01$ measure the absolute change in the value of the bond for a fall of one basis point in market interest rates.

value are a linear function of changes in the risk factors. For this reason we consider a linear portfolio[2].

The risk and return on the portfolio we estimate as follows (Alexander, 2008b). Let $\boldsymbol{C} = (C_{T_1}, \ldots, C_{T_n})^T$ is a mapped cash flow vector, not expressed in present value terms and $\boldsymbol{R} = (R_{T_1}, \ldots, R_{T_n})^T$ is discretely compounded market interest rates at these maturities. Suppose that the interest rate R_{T_1} of maturity T_i changes by a small amount ΔR_{T_1} [3], say, a few basis points. The $PV01$ at this maturity is the increase in present value of the cash flow when the interest rate falls by one basis point. So we can approximate the change in present value of the mapped cash flow at maturity T_i as $\Delta PV \approx -PV01_{T_i} \times \Delta R_{T_i}$. Now suppose all interest rates changes, but they can change by different amounts, and denote the vector of interest rate changes (in basis points) as $\Delta\boldsymbol{R} = (\Delta R_{T_1}, \ldots, \Delta R_{T_n})^T$. Then the portfolio's Profit & Losses (P&L) is the net change in present value of the entire cash flow, i.e.

$$\Delta PV \approx -\sum_{i=1}^{n} PV01_{T_i} \times \Delta R_{T_i}, \tag{1}$$

or, in matrix notation,

$$\Delta PV \approx \boldsymbol{p}^T \Delta \boldsymbol{R}, \tag{2}$$

where $\boldsymbol{p}=(PV01_{T_1}, \ldots, PV01_{T_n})^{\mathrm{T}}$.

Hence knowing the risk factor sensitivities, i.e. the $PV01$ vector $\boldsymbol{p}$, we have an expression for the discounted P&L of the portfolio as a linear function of any absolute changes in interest rates. Furthermore, if $\Delta\boldsymbol{R}$ has mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{V}$ then, based on the linear approximation (2), the mean of the discounted P&L is $-\boldsymbol{p}^T\boldsymbol{\mu}$ and its variance is $\boldsymbol{p}^T\boldsymbol{V}\boldsymbol{p}$. Since the returns $\Delta\boldsymbol{R}$ are measured in basis points we also express their volatilities and correlations in basis points.

Now we perform a PCA on the covariance matrix $\boldsymbol{V}$ of the changes in the interest rates and obtain a principal component approximation for each interest rate change:

$$\Delta R_{T_i} \approx w_{i,1} P_{T_1} + \ldots + w_{i,k} P_{T_k}, \tag{3}$$

[2] All the non–linearity in the relationship between portfolio value and the risk factors is subsumed into the risk factor sensitivities. These sensitivities are held constant when we study the risk of the portfolio. Our portfolio contains no options.

[3] Δ symbolizes the difference operator

where P_{T_j} is the value of the j–th principal component at time T_j, w_{ij} is the i–th element of the j–th eigenvector of $\boldsymbol{V}$ and k is small compared with n (k is usually taken to be 3 or 4). The j–th principal component risk factor sensitivity is then given by

$$w_j = \sum_{i=1}^{n} PV01_i \, w_{i,j} \, . \tag{4}$$

It means if we take the dot product of the vector $\boldsymbol{p}$ and the j–th eigenvector $\boldsymbol{w}_j$ we obtain the j–th principal component risk factor sensitivity.

Now substituting (3) into (2) and using (4) yields the principal component factor model representation of the portfolio P&L as

$$\Delta PV \approx \boldsymbol{w}^T \boldsymbol{p}_T \tag{5}$$

where the $k \times 1$ vectors $\boldsymbol{p}_T = (P_{T_1}, \ldots, P_{T_k})^T$ and $\boldsymbol{w} = (w_1, \ldots, w_k)^T$ denote the principal component risk factors at time T, and their (constant) factor sensitivities. Comparing (2) with (5), the number of risk factors has been reduced from n to k.

The variance of the ΔPV is then calculated as

$$V(\Delta PV) \approx \boldsymbol{w}^T V(\boldsymbol{p}_T)\boldsymbol{w}, \tag{6}$$

where $V(\boldsymbol{p}_T)$ is covariance matrix of the principal component.

Fixed income portfolios typically have an extremely large number of highly correlated risk factors. But PCA allows us to reduce the dimension of the risk factor space from, for instance, n=60 to k=3. Moreover, the principal component risk factors have an intuitive interpretation: the first component captures an approximately parallel shift in the entire yield curve, and the second and third components capture a change in slope and a change in curvature of the yield curve. Together, first three components often explain over 95% of the variation in interest rates in major currencies such as the US dollar, Euro or British pound, but less in emerging currencies where the fixed income markets are less liquid and so the correlation between interest rates is lower. The amount of risk factor variation explained by the first three or four principal components depends on the frequency of the interest changes: weekly and monthly changes are usually more highly correlated than daily changes, so a larger fraction of the total variation can be explained by the first few components.

Now we use this model for decomposing the basic risk of a forward position in a foreign currency.

The main component of basis risk in currency forwards is the fluctuation of the forward price around its fair or theoretical price, which is based on the spot price. In liquid currency markets the forward price is normally very close to its fair price so the basis risk is negligible. In this case we can model currency forwards by decomposing each forward exposure into a spot exposure and an exposure to the risk free zero coupon interest rate differential of the same maturity as the forward (Alexander, 2008a).

We suppose that at time T_i we have a sequence of cash payments $\boldsymbol{C}$ in a foreign currency. Denote by ΔPV^d the change in present value at time T_i of the entire sequence of cash flows in domestic currency when the domestic interest rate of maturity T_i change by amounts $\Delta R_{T_i}^d = \left(\Delta R_{T_1}^d, \ldots, \Delta R_{T_n}^d\right)^T$. From (1) ΔPV^d is the sum of the present values of all the cash flows:

$$\Delta PV^d \approx -\sum_{i=1}^{n} PV01_{T_i}^d \times \Delta R_{T_i}^d, \tag{7}$$

where $PV01_{T_i}^d$ is the PV01 sensitivity of the cash flow in domestic currency at maturity T_i.

Similarly, we use (1) for the change in present value of the sequence of cash flows in foreign currency ΔPV^f when the domestic interest rates change by amounts $\Delta R_{T_i}^d$ is defined as

$$\Delta PV^f \approx -\sum_{i=1}^{n} PV01_{T_i}^f \times \Delta R_{T_i}^d, \tag{8}$$

Now let S_t denotes the domestic foreign exchange rate at time T_{i}, then

$$\Delta PV^d = S_t \Delta PV^f \tag{9}$$

It can be shown that (9) implies

$$R_{T_i}^d \approx R_{T_i}^S + R_{T_i}^f \tag{10}$$

where $R_{T_i}^d$ is the return on the cash flow in domestic currency, $R_{T_i}^f$ is the return on the cash flow in foreign currency and $R_{T_i}^S$ is the return on the spot exchange rate.

Using the approximation (10), we may decompose the risk on a sequence of foreign currency forward payments into exchange rate and interest rate risks and correlation risk. Taking variances of (10) yields the risk decomposition

$$V(R_{T_i}^d) \approx V(R_{T_i}^S) + V(R_{T_i}^f) + 2\operatorname{cov}(R_{T_i}^S, R_{T_i}^f). \tag{11}$$

However, although the exchange rate risk is defined in terms of the variance of returns, the interest rate risk from a PCA factor model is defined in terms of the variance of the P&L and not the variance of returns. So we rewrite (11) in a form that can be applied, i.e.

$$V\left(\Delta P^d\right) = \left(\overline{P}^d\right)^2 V\left(R_{T_i}^S\right) + \left(\overline{S}\right)^2 V\left(\Delta PV^f\right) + 2\overline{P}^f\,\overline{S}\operatorname{cov}\left(R_{T_i}^S, \Delta PV^f\right) \tag{12}$$

where $\overline{P}^d$ and $\overline{P}^f$ are the present values of the cash flows in domestic and foreign currencies respectively, and $\overline{S}$ is the exchange rate at the time that the risk is measured (they are fixed values).

In this situation a PCA factor model of the interest rates gives us the possibility to estimate these terms very precisely using only three components.

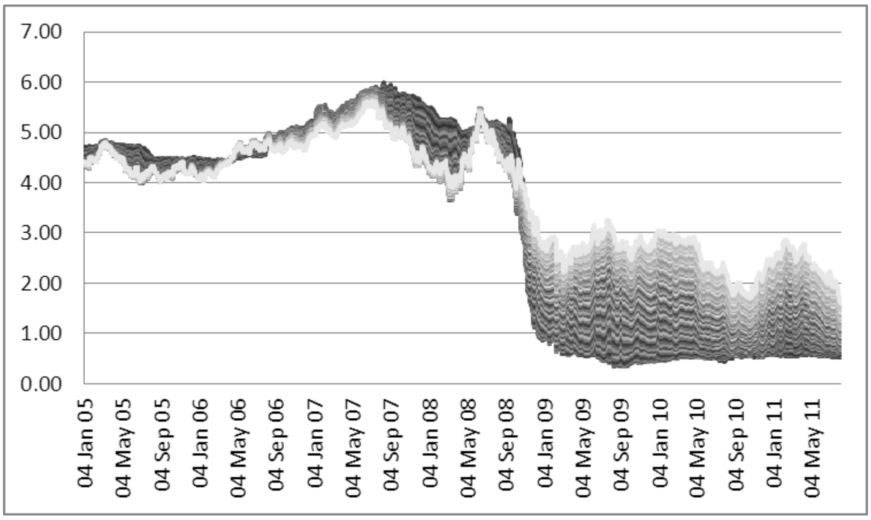
2 Analysis of the risk for forward currency exposure

In this paper we analysed the risk UK company, that has the forward payments of 1 million EUR on the 5th of every month over the next 5 years. We have found key risk factors of exposures and we have used PCA factor model to the UK short spot rate to describe the interest rate, foreign exchange and correlation risk on 4 august 2011.

The sources of dataset are Bank of England's interactive statistical database and European Central Bank. We used the Bank of England daily interest rate data from 1 month to 60 months between 4 January 2005 and 4 August 2011 and the daily exchange rate GBP/EUR data to over the same period.

Fig. 1 illustrates the UK government interest rate of selected maturities for the whole zero coupon curve from January 4, 2005 to August 4, 2011. Interest rates were from 0.34% to 6%. The period since January 4 2005 until August 2007 was characterized by increasing spot rate curve. In August 2007 spot rates were high. After August 2007 spot rate curve has descended. Next period is characterized by humped spot rate curve. Since Jun 2008 spot rate has begun to fall. The fall of interest rates produce fuel economic growth.

Fig. 1: UK government interest rate, daily data for maturity from 1 to 60 month, 4.1.2005-4.8.2011



Source: Calculated by the authors with Wolfram Mathematica software on base data from Bank of England,

Yield curves form a highly collinear system. To aid PCA we extracted three uncorrelated time series from this system to use in a subsequent analysis of the risk. A PCA has been performed on the 60×60 daily covariance matrix calculated from daily changes in the short spot curve between 4 January 2005 and 4 August 2011. The PCA output is the eigenvalues and eigenvectors of this matrix. The eigenvalues are all positive because the matrix is positive definite, and they have been ordered in decreasing order of magnitude. Tab. 1 summarizes the first nine eigenvalues, ordered from largest to smallest, percentage variation explained and cumulative variation explained.

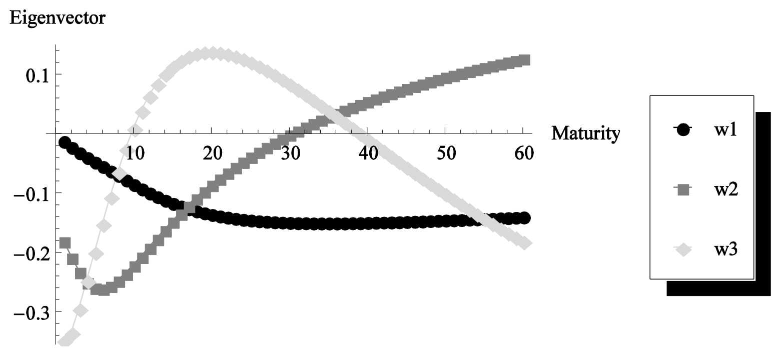
The first eigenvalue is 1294.46, the first principal component explains 92.83% of the covariance between changes in UK short spot rates. The second eigenvalue is 69.14, the second principal component explains 4.95% of the variation in the system and that, taken together, the first two principal components explain 97.79% of the covariance between changes in UK short spot rates. The third eigenvalue is 22.26, the third principal component explains 1.59% of the variation in the system and that, taken together, the first three principal components explain 99.82% of the covariance between changes in UK short spot rates. If we add the fourth principal components to represent the system, we can explain 99.9% of the covariance between changes in UK short spot rates. This finding is typical of any highly correlated term structure, although of course the exact results will depend on the series used, its frequency and the data period chosen. From this follows, we can use only first three eigenvectors for risk calculations. We have drawn them as a function of maturity in Fig. 2.

Tab. 1: Eigenvalues of the covariance matrix for UK short spot rate

Component	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Eigenvalue	1294.46	69.14	22.26	5.97	1.97	0.40	0.10	0.01	0.00
% Variation	92.83%	4.95%	1.59%	0.04%	0.01%	0.00%	0.00%	0.00%	0.00%
Cumulative %	92.83%	97.79%	99.82%	99.99%	99.99%	99.99%	99.99%	99.99%	100%

Source: Calculated by the authors with Wolfram Mathematica software on base data from Bank of England

Fig. 2: First three eigenvectors of the UK daily short spot rate covariance matrix



Source: Calculated by the authors with Wolfram Mathematica software on base data from Bank of England

The first eigenvector (labelled w1) is almost a horizontal line, because the first eigenvector has almost identical values on each maturity. This means that if the first principal component shifts upwards, leaving the other principal components fixed then all the spot rates will move upwards in an approximately parallel shift. From eigenvalue analysis follows that this type of parallel shift accounts for 92.83% of the movements in short spot rates during 2005–2011. The second eigenvector (labelled w2) is a monotonic decreasing function of maturity. An upward shift in the second component, leaving the other components fixed, induces a tilt in the forward curve, with an upward move at the short end and a downward move at the long end. This type of movement accounts for nearly 4.95% of the variation short spot rates during 2005–2011. The third eigenvector (labelled w3) has shape similar to a quadratic function of maturity, being highest at the approximately 20 month maturity and with growing maturity go down. The third eigenvector is negative at the short (maturity less than 10 months) end and the long end (maturity greater than 40 months) and positive for middle maturities (between 10 and 40 months). An upward shift in the third principal component (leaving the other components fixed) will change the convexity of the interest rate curve. It will make a downward sloping curve more convex and an upward sloping curve less convex. This type of movement accounts for only 1.59% of the variation in short spot rates during 2005–2011.

These eigenvectors represent key risk factors for exposures our payments. The PCA factor model (5) that approximates the change in present value (P&L) of the sequence of Euro currency cash flows when the UK interest rates change[4]:

[4] We take the dot product between the $PV01^{EUR}$ vector and the i–th eigenvector to get the net weight on the i–th principal component, for $i=1,2$ and 3.

$$\Delta P_T^{EUR}{}_t \approx 2026.18 \times P_{T_1} - 541.75 \times P_{T_2} + 430.91 \times P_{T_3}, \qquad (13)$$

where P_{T_1}, P_{T_2} and P_{T_3} are the first three principal components.

Now we decompose the risk of foreign payments into the interest rate, foreign exchange and correlation risk on 4 august 2011. Firstly, we computed total present value of the foreign payments based on GBP discount curve and EUR discount curve (see Tab. 2).

Tab. 2: Total present value of the forward payments

Total PV based on UK discount curve (in GBP)	50624923.669 GBP
Total PV based on EUR discount curve (in EUR)	59983158.378 EUR
Total PV based on EUR discount curve (equivalent in GBP)	52023554.534GBP

Source: Calculated by the authors with Wolfram Mathematica software on base data from Bank of England and European Central Bank

Tab. 3: Volatilities of the forward payments

Historical FX Volatility	8.7%
Volatility P&L in EUR	73066.391EUR
Volatility P&L in GBP	63370.677 GBP

Source: Calculated by the authors with Wolfram Mathematica software on base data from Bank of England and European Central Bank

From daily historical data of the GBP/EUR exchange rate has been obtained annualized historical FX volatility 8.7%. The calculation of the risk of P&L (ΔPV^{EUR}) follows from (6) (we use the current GBP/EUR exchange rate 1.153), see Tab. 3.

Interest rate risk has been obtained taking the volatility of P&L, annualizing it (using $\sqrt{252}$) and multiplying by current exchange rate. FX risk has been obtained multiplying the present value of the payments in GBP by the exchange rate volatility. The last component of the risk decomposition (12) is the correlation risk. This is represented by the term corresponding to the covariance between UK interest rate and exchange rates i.e. $2\overline{P}^{EUR}(GBP/EUR_{current})\operatorname{cov}\left(R_{T_i}^S, \Delta PV^{EUR}\right)$. $\overline{P}^{EUR}$ is present value of the payments in EURO based on the EURO discount curve. For computing of the $\operatorname{cov}\left(R_{T_i}^S, \Delta PV^{EUR}\right)$ has been used PCA factor model (13). Tab. 4 summarizes these results.

Tab. 4: Summary of the decomposing of the risks for the forward payments

IR Risk	1005978.316 GBP
FX Risk	4395458.370 GBP

Correlation risk	1590120.894GBP
Total risk	4781268.777 GBP

Source: Calculated by the authors with Wolfram Mathematica software on base data from Bank of England and European Central Bank

From Tab. 4 is see that interest rate and correlation risks are negligible compared with the FX risk.

Conclusion

The PCA factor model for forward payments was used in this paper with purpose to measure the interest rate risk, exchange risk and correlation risk. The first part of this paper shows how Principal Component method can be used for decomposing of the risk of the forward payments. The second part of this paper presents an empirical model for decomposing the risk, based on three key risk factors. The analysis of the risk is greatly simplified, because it is based on only few risk factors. Without the PCA factor model, the decomposition of the total risk in practice would be more complex computationally and also very ad hoc. Since the key risk factors are independent, PCA factor model allows us to estimate of the interest rates risk, exchange rate risk and correlation risk precisely and computational efficient.

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